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Refined BPS state counting from Nekrasov's formula and Macdonald functions

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Abstract

It has been argued that Nekrasov's partition function gives the generating function of refined BPS state counting in the compactification of M theory on local Calabi-Yau spaces. We show that a refined version of the topological vertex we previously proposed (hep-th/0502061) is a building block of Nekrasov's partition function with two equivariant parameters. Compared with another refined topological vertex by Iqbal, Kozcaz and Vafa (hep-th/0701156), our refined vertex is expressed entirely in terms of the specialization of the Macdonald symmetric functions which is related to the equivariant character of the Hilbert scheme of points on $\mathbb{C}^2$. We provide diagrammatic rules for computing the partition function from the web diagrams appearing in geometric engineering of Yang-Mills theory with eight supercharges. Our refined vertex has a simple transformation law under the flop operation of the diagram, which suggests that homological invariants of the Hopf link are related to the Macdonald functions.

1 Introduction

The problem of instanton counting is one of the important aspects of nonperturbative dynamics in gauge and string theory. The result is encoded in the partition function of topological gauge and string theories, which is often computed exactly by the duality and/or the localization principle. A celebrated example in gauge theory is Nekrasov's partition function $Z_{Nek}(\epsilon_i, a_\ell, \Lambda)$, which reproduces the Seiberg-Witten prepotential from the microscopic viewpoint of equivariant integration over the instanton moduli space [1]. On the string theory side, the topological vertex $C_{\lambda_1\lambda_2\lambda_3}(q)$ is constructed based on the geometric transition, which is a duality of topological closed string to the Chern-Simons theory [2, 3]. The topological vertex provides a building block of all genus topological string amplitudes on local toric Calabi-Yau 3-fold. It is amusing that these two instanton counting problems are actually related in an appropriate setup, which is expected from geometric engineering [4, 5].

To compute the integration over the instanton moduli space of $SU(N)$ gauge theory, Nekrasov considered the toric action on $\mathbb{R}^4 \simeq \mathbb{C}^2 \ni (z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$, which induces the toric action on the moduli space of framed instantons. By the localization theorem, the integral becomes a sum over the contributions from each fixed point of the toric action, which is labeled by the set of N Young diagrams (the "colored" partitions) $\{\lambda_\ell\}_{\ell=1}^N$, $(\lambda_{\ell,1} \geq \lambda_{\ell,2} \geq \cdots \geq \lambda_{\ell,i} \geq \lambda_{\ell,i+1} \geq \cdots)$. As a consequence we obtain Nekrasov's partition function [6]:

$$Z_{Nek}(\epsilon_1, \epsilon_2, a_\ell, \Lambda) = \sum_{\{\lambda_\ell\}} \frac{\Lambda^{2N|\lambda|}}{\prod_{\alpha,\beta=1}^N n_{\alpha,\beta}^{\{\lambda_\ell\}}(\epsilon_1, \epsilon_2, a_\ell)} \ , \tag{1.1}$$

where $|\lambda| = \sum_{\ell=1}^N \sum_{1\leq i} \lambda_{\ell,i}$ and

$$\begin{aligned} n_{\alpha,\beta}^{\{\lambda_\ell\}} = &\prod_{s\in\lambda_\alpha} \left(-\ell_{\lambda_\beta}(s)\epsilon_1 + (a_{\lambda_\alpha}(s)+1)\epsilon_2 + a_\beta - a_\alpha\right) \\ &\prod_{t\in\lambda_\beta} \left((\ell_{\lambda_\alpha}(t)+1)\epsilon_1 - a_{\lambda_\beta}(t)\epsilon_2 + a_\beta - a_\alpha\right) . \end{aligned} \tag{1.2}$$

The parameter Λ of instanton expansion is introduced as a dynamical scale in the renormalization. The vacuum expectation values of the scalar fields in the vector multiplets are $a_\alpha, 1 \leq \alpha \leq N$. Mathematically they are equivariant parameters for the action of maximal torus on the gauge group. We denote the leg length and the arm length at $s = (i, j)$ with respect to the Young diagram λ by $\ell_\lambda(i,j) = \lambda_j^\vee - i$, and $a_\lambda(i,j) = \lambda_i - j$, respectively. The relation to the (topological) string theory becomes transparent, if we

consider a five-dimensional ("trigonometric", or K-theoretic) lift of the partition function by promoting the factors $n^{\{\lambda_\ell\}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, a_\ell)$ in the denominator to

$$N^{\{\lambda_\ell\}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, a_\ell) = \prod_{s\in\lambda_\alpha} \left(1 - t^{-\ell_{\lambda_\beta}(s)} q^{-a_{\lambda_\alpha}(s)-1} \mathbf{e}_\beta/\mathbf{e}_\alpha\right) \prod_{t\in\lambda_\beta} \left(1 - t^{\ell_{\lambda_\alpha}(t)+1} q^{a_{\lambda_\beta}(t)} \mathbf{e}_\beta/\mathbf{e}_\alpha\right), \tag{1.3}$$

where $(q,t) := (e^{\epsilon_2}, e^{-\epsilon_1})$ and $\mathbf{e}_\alpha := e^{-a_\alpha}$. We can show that, when $q = t = e^{-g_s}$, Nekrasov's partition function is nothing but the topological string amplitude on an appropriate local toric Calabi-Yau manifold [7]–[12].

All genus topological string amplitude on local Calabi-Yau 3-fold can be computed by a diagrammatic rule, in terms of the topological vertex:

$$C_{\mu\lambda\nu}(q) = q^{\frac{\kappa(\nu)}{2}} s_\lambda(q^\rho) \sum_\eta s_{\mu/\eta}(q^{\lambda^\vee+\rho}) s_{\nu^\vee/\eta}(q^{\lambda+\rho}) , \tag{1.4}$$

where $s_{\lambda/\mu}(x)$ is the (skew) Schur function and $q^{\lambda+\rho}$ means the substitution $x_i := q^{\lambda_i - i + \frac{1}{2}}$. The partition $\lambda^\vee$ is defined by the transpose of the corresponding Young diagram. The definition of $\kappa(\lambda)$ is given in Appendix E[1]. Then a natural question is: For generic parameters (ϵ_1, ϵ_2), can we obtain $Z_{Nek}(\epsilon_i, a_\ell, \Lambda)$ in a similar manner by generalizing the topological vertex $C_{\mu\lambda\nu}(q)$? This is the problem of constructing a refined topological vertex. An answer to this question has been given by Iqbal, Kozcaz and Vafa [13]. The refined topological vertex they proposed is

$$C^{(IKV)}_{\mu\nu\lambda}(t,q) = \\ \left(\frac{q}{t}\right)^{\frac{||\nu||^2+||\lambda||^2}{2}} t^{\frac{\kappa(\nu)}{2}} P_{\lambda^\vee}(t^{-\rho}; q, t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^\vee/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^\vee} q^{-\rho}) . \tag{1.5}$$

As before, $q^\lambda t^\rho$ etc. means the specialization $x_i := q^{\lambda_i} t^{\frac{1}{2}-i}$. On the other hand, before the proposal in [13] we had introduced the following vertex in [14][2]:

$$C_{\mu\lambda}{}^{\nu}(q,t) = \\ f_\nu(q,t)^{-1} P_\lambda(t^\rho; q, t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|-|\nu|}{2}} \iota P_{\mu^\vee/\eta^\vee}(-t^{\lambda^\vee} q^\rho; t, q) P_{\nu/\eta}(q^\lambda t^\rho; q, t) . \tag{1.6}$$

It is convenient to introduce the conjugate vertex $C^{\mu\lambda}{}_{\nu}(q,t) := C_{\mu^\vee\lambda^\vee}{}^{\nu^\vee}(t,q)(-1)^{|\mu|+|\lambda|+|\nu|}$. Note that the conjugation involves the exchange of t and q. The refined vertex $C^{(IKV)}_{\mu\nu\lambda}(t,q)$

[1]Our notations for partitions are summarized in Appendix E.

[2]We have slightly changed the original definition in [14] by improving the framing factor.

partly employs the Macdonald function $P_\lambda(x;q,t)$ [15], but there still remain the skew Schur functions[3]. Compared with it, our proposal eliminates the skew Schur functions completely and the vertex is expressed in terms of the (skew) Macdonald function $P_{\lambda/\mu}(x;q,t)$. The price for this elimination is that we have to introduce the involution ι on the algebra of symmetric functions defined by $\iota(p_n) = -p_n$, where $p_n(x) := \sum_{i=1}^{\infty} x_i^n$ is the power sum function. Since $\{p_n(x)\}_{n=1}^{\infty}$ forms a "multiplicative" basis of the algebra of the symmetric functions, the involution ι is uniquely defined by the above relation. Finally $f_\lambda(q,t) := (-1)^{|\lambda|} q^{\frac{||\lambda||^2}{2}} t^{-\frac{||\lambda^\vee||^2}{2}}$ is the framing factor proposed recently by Taki [16]. The refined vertex $C^{(IKV)}_{\mu\nu\lambda}(t,q)$ has a nice interpretation as the counting of "unisotropic" plane partitions, or by statistical mechanics of the melting crystal model [17, 13, 18]. Although the relation of our vertex to such a statistical model is unclear, our vertex is more symmetric than $C^{(IKV)}_{\mu\nu\lambda}(t,q)$, since we have replaced all the (skew) Schur functions in the topological vertex by the (skew) Macdonald functions. However, it seems impossible to make $C_{\mu\lambda}{}^{\nu}(q,t)$ completely symmetric under the cyclic permutation of partitions.

In [13] it is claimed (see also the arguments in [16]) that one can reproduce Nekrasov's partition function from the refined topological vertex $C^{(IKV)}_{\mu\nu\lambda}(t,q)$. As we mentioned already in [14], our vertex (1.6) also reproduces the $SU(N)$ Nekrasov's partition function, and in this article we will show it concretely. Though $C^{(IKV)}_{\mu\nu\lambda}(t,q)$ and $C_{\mu\lambda}{}^{\nu}(q,t)$ are different, they give the same result as long as we put trivial representations to external edges, which is the case when we compute Nekrasov's partition function by the method of topological vertex. The Schur functions and the Macdonald functions are two different basis of the space of symmetric functions and hence they satisfy the Cauchy formulas of the same type (see section 5.2 and Appendix B). This is a technical reason why $C^{(IKV)}_{\mu\nu\lambda}(t,q)$ and $C_{\mu\lambda}{}^{\nu}(q,t)$ give the same result after taking the summation over the partitions attached to internal edges. We should emphasize that, throughout this article except for appendix D, q and t are treated as formal parameters and we do not use any asymptotic computations such as $\lim_{N\to\infty} q^N = 0$ otherwise stated.

In this paper we show that the refined topological vertex gives a building block of the K-theoretic lift of Nekrasov's partition function. We would like to point out the following possible application. It has been argued Nekrasov's partition function gives the generating function of refined BPS state counting in the compactification of M theory on local Calabi-Yau spaces [11, 14, 13]. As far as we know (see appendix C, for examples), the refined BPS state counting always gives integers, which is a refined version of the conjecture of the integrality of the Gopakumar-Vafa invariants [19]–[21]. Recently the

[3] However, this implies a nice interpretation in terms of plane partitions.

conjecture has been proven for local toric Calabi-Yau 3-folds [22]–[24]. The existence of the topological vertex is one of the important ingredients in the proofs. Hence one may expect that the refined vertex is helpful in proving the integrality of the refined Gopakumar-Vafa invariants for the local toric case. Macdonald functions are also related to q-deformed Virasoro and $\mathcal{W}$ algebras [25]. We hope these quantum groups play an important role in topological string theory and Yang-Mills theory.

The paper is organized as follows: In section 2 we introduce the K-theoretic lift of Nekrasov's partition function following a mathematical formulation in [26, 27]. The K-theoretic lift allows the Chern-Simons coupling $m \in \mathbb{Z}$ and we find that the framing factor $f_\lambda(q,t)$ of the refined topological vertex arises naturally from the m dependence of the partition function. We also examine the symmetry of the partition function under $r_L : (q,t) \to (q^{-1}, t^{-1})$ and $r_R : (q,t) \to (t,q)$. This is a necessary condition for the K-theoretic lift to be interpreted as a character of $Spin(4) = SU(2)_L \times SU(2)_R$. In section 3 we review the idea of geometric engineering. When we compute the partition function using the refined topological vertex as a building block, we will fix a preferred direction, for which we mainly choose the horizontal left arrow $(-1,0)$ in this paper. The (dual) toric diagram in geometric engineering has a feature in which we can arrange the diagram so that each vertex has a unique edge with the preferred direction. To emphasize the fact that we fix the preferred direction of the diagram, we will call it web diagram in the following. We define our refined topological vertex in section 4. The gluing rules of the vertex for computing the partition function are also provided. In section 5 we consider four-point functions obtained by gluing two refined vertices. We show that the four-point function enjoys a rather simple transformation law under the flop operation of the web diagram. Based on this transformation law we argue a possible relation of our refined vertex to homological invariants of the Hopf link [28, 29]. In section 6 we discuss one-loop diagrams with some examples. Finally, we present several examples of the computation of the partition function in sections 7–9. In appendix A we explain the equivalence between several expressions of the Nekrasov formula. In appendix B we give a definition of the Macdonald symmetric functions and collect several useful formulas. We present examples of the refined BPS state counting in appendix C. In appendix D we remark that our refined topological vertex can be expressed in terms of the q-Dunkl operator. Appendix E gives a list of notations and some identities for partitions used in this paper.

The following notations are used through this article. q and t are formal parameters otherwise stated. Let λ be a Young diagram, i.e. a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, which is

a sequence of nonnegative integers such that $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| = \sum_i \lambda_i < \infty$. $\lambda^\vee$ is its conjugate (dual) diagram. $\ell(\lambda) = \lambda_1^\vee$ is the length and $|\lambda| = \sum_i \lambda_i$ is the weight. For each square $s = (i, j)$ in λ,

$$\begin{aligned} a(s) &:= \lambda_i - j, & a'(s) &:= j - 1, \\ \ell(s) &:= \lambda_j^\vee - i, & \ell'(s) &:= i - 1, \end{aligned} \tag{1.7}$$

are the arm length, arm colength, leg length and leg colength, respectively[4]. Let $p_n(x) = \sum_{i=1}^\infty x_i^n$ be the power sum function in the set of variables $x = (x_1, x_2, \cdots)$. If $|t^{-1}| < 1$, the variable $q^\lambda t^\rho$ stands for $x_i = q^{\lambda_i} t^{\frac{1}{2}-i}$. But for all $t \in \mathbb{C}$, we define

$$\begin{aligned} p_n(cq^\lambda t^\rho) &:= c^n \sum_{i=1}^\infty (q^{n\lambda_i} - 1) t^{n(\frac{1}{2}-i)} + \frac{c^n}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}, \qquad q, t, c \in \mathbb{C}, \\ p_n(cq^\lambda t^\rho, cLt^{-\rho}) &:= c^n \sum_{i=1}^\infty (q^{n\lambda_i} - 1) t^{n(\frac{1}{2}-i)} + c^n \frac{1 - L^n}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}, \qquad q, t, c, L \in \mathbb{C} \\ &= c^n \sum_{i=1}^N q^{n\lambda_i} t^{n(\frac{1}{2}-i)}, \qquad q, t, c \in \mathbb{C}, \quad L = t^{-N}, N \in \mathbb{N}. \end{aligned} \tag{1.8}$$

All symmetric functions in this article are treated as polynomials in the power sum symmetric functions $(p_1, p_2, \cdots)$. Finally, we often use $u := (qt)^{\frac{1}{2}}$ and $v := (q/t)^{\frac{1}{2}}$.

2 Structure of Nekrasov's Partition Function

2.1 Partition function with Chern-Simons coupling

The five-dimensional lift of Nekrasov's partition function of $SU(N_c)$ theory is given by the summation over the set of N_c Young diagrams (or colored partitions) $\{\lambda_\alpha\}_{\alpha=1}^{N_c}$, as follows:

$$Z^{\text{inst}}(\epsilon_1, \epsilon_2; a_\alpha, \Lambda) = \sum_{\{\lambda_\alpha\}} \frac{\left(e^{-\frac{\epsilon_1+\epsilon_2}{2}} \Lambda^2\right)^{N_c \cdot |\lambda|}}{\prod_{\alpha,\beta=1}^{N_c} N_{\alpha,\beta}^{\{\lambda_\alpha\}}(\epsilon_1, \epsilon_2; a_\alpha)}.$$

The product in the denominator is the equivariant Euler character of the tangent space to the instanton moduli space $M(N_c, k)$ at a fixed point of the toric action, which is labeled by $\{\lambda_\alpha\}_{\alpha=1}^{N_c}$ with $|\lambda| = k$. Each factor is given by

$$N_{\alpha,\beta}^{\{\lambda_\alpha\}}(\epsilon_1, \epsilon_2; a_\alpha) = \prod_{s \in \lambda_\alpha} \left(1 - e^{\ell_{\lambda_\beta}(s)\epsilon_1 - (a_{\lambda_\alpha}(s)+1)\epsilon_2 + a_\alpha - a_\beta}\right) \prod_{t \in \lambda_\beta} \left(1 - e^{-(\ell_{\lambda_\alpha}(t)+1)\epsilon_1 + a_{\lambda_\beta}(t)\epsilon_2 + a_\alpha - a_\beta}\right). \tag{2.1}$$

[4]In [13] the definitions of the arm length and the leg length are exchanged.

The product $\prod_{\alpha,\beta=1}^{N_c} N_{\alpha,\beta}^{\{\lambda_\alpha\}}(\epsilon_1,\epsilon_2;a_\alpha)$ consists of $2N_c k$ factors, which agree to the complex dimensions of $M(N_c,k)$. In [26, 27] the five-dimensional lift is mathematically identified as the K-theoretic lift and it is computed as follows:

$$\begin{aligned}
&Z_m^{\text{inst}}(\epsilon_1,\epsilon_2;a_\alpha,\Lambda)\\
&=\sum_{k=0}^{\infty}\left(e^{-\frac{1}{2}(N_c+m)(\epsilon_1+\epsilon_2)}\Lambda^{2N_c}\right)^k\sum_i(-1)^i\mathrm{ch}H^i(M(N_c,k),\mathcal{L}^{\otimes m})\\
&=\sum_{\{\lambda_\alpha\}}\frac{\left(e^{-\frac{1}{2}(N_c+m)(\epsilon_1+\epsilon_2)}\Lambda^{2N_c}\right)^{|\lambda|}}{\prod_{\alpha,\beta}N_{\alpha,\beta}^{\{\lambda_\alpha\}}(\epsilon_1,\epsilon_2;a_\alpha)}\cdot\exp\left(m\sum_\alpha\sum_{s\in\lambda_\alpha}(a_\alpha-\ell'(s)\epsilon_1-a'(s)\epsilon_2)\right)\;, \quad (2.2)
\end{aligned}$$

where $M(N_c,k)$ is the framed moduli space of rank N_c torsion free sheaves E on $\mathbb{P}^2$ with $c_2(E)=k$. The line bundle $\mathcal{L}$ over $M(N_c,k)$ is defined by

$$\mathcal{L}:=\det\left[R^1(p_2)_*(\mathcal{E}\otimes(p_1)^*\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))\right]\;, \tag{2.3}$$

where $\mathcal{E}$ is the universal sheaf on $\mathbb{P}^2\times M(N_c,k)$ and $p_{1,2}$ is the projection to the first or the second component. Physically Z_m^{inst} is the instanton part of the partition function of $SU(N_c)$ gauge theory on $\mathbb{R}^4\times S^1$ with eight supercharges, and the power $m\in\mathbb{Z}$ of the line bundle $\mathcal{L}$ is identified as the coefficient of the five-dimensional Chern-Simons term [30].

Let $(q,t):=(e^{\epsilon_2},e^{-\epsilon_1})$, $\mathbf{e}_\alpha:=e^{-a_\alpha}$ and $Q_{\alpha,\beta}:=\mathbf{e}_\alpha/\mathbf{e}_\beta$;[5] then $Z_m^{\text{inst}}(\epsilon_1,\epsilon_2;a_\alpha,\Lambda)$ is written as

$$Z_m^{\text{inst}}(\mathbf{e}_1,\cdots,\mathbf{e}_{N_c},\Lambda;q,t)=\sum_{\{\lambda_\alpha\}}\frac{\prod_{\alpha=1}^{N_c}\left(v^{-N_c}\Lambda^{2N_c}(-\mathbf{e}_\alpha)^{-m}\right)^{|\lambda_\alpha|}f_{\lambda_\alpha}(q,t)^{-m}}{\prod_{\alpha,\beta=1}^{N_c}N_{\lambda_\alpha\lambda_\beta}(Q_{\alpha,\beta};q,t)}\;, \tag{2.4}$$

with $v:=(q/t)^{\frac{1}{2}}$ and

$$\begin{aligned}
N_{\lambda_\alpha\lambda_\beta}(Q_{\alpha,\beta};q,t)&:=N_{\beta,\alpha}^{\{\lambda_\alpha\}}(\epsilon_1,\epsilon_2;a_\alpha)\\
&=\prod_{s\in\lambda_\alpha}\left(1-q^{a_{\lambda_\alpha}(s)}t^{\ell_{\lambda_\beta}(s)+1}Q_{\alpha,\beta}\right)\prod_{t\in\lambda_\beta}\left(1-q^{-a_{\lambda_\beta}(t)-1}t^{-\ell_{\lambda_\alpha}(t)}Q_{\alpha,\beta}\right), \quad (2.5)
\end{aligned}$$

where $|\lambda|$ is the number of boxes of λ and

$$f_\lambda(q,t):=\prod_{s\in\lambda}(-1)q^{a'(s)+\frac{1}{2}}t^{-\ell'(s)-\frac{1}{2}}=\prod_{(i,j)\in\lambda}(-1)q^{\lambda_i-j+\frac{1}{2}}t^{-\lambda_j^\vee+i-\frac{1}{2}}, \tag{2.6}$$

[5] The different conventions $(q,t):=(e^{-\epsilon_2},e^{\epsilon_1})$ and $Q_{\beta,\alpha}:=\mathbf{e}_\alpha/\mathbf{e}_\beta$ are also used in the literature.

is the framing factor [6] which has been proposed by Taki [16]. This is nothing but the m dependent (q,t) factor of the partition function. Note that the framing factor satisfies the following symmetry:

$$f_\lambda(q,t) = f_\lambda\left(q^{-1},t^{-1}\right)^{-1} = f_{\lambda^\vee}(t,q)^{-1}. \tag{2.7}$$

Let $u=(qt)^{\frac{1}{2}}$ and $v=(q/t)^{\frac{1}{2}}$. We have the following six equivalent expressions of $N_{\lambda\mu}(Q;q,t)$:

Proposition.

$$N_{\lambda\mu}(Q;q,t) = \prod_{(i,j)\in\mu}\left(1-Q\,q^{\lambda_i-j}t^{\mu_j^\vee-i+1}\right)\prod_{(i,j)\in\lambda}\left(1-Q\,q^{-\mu_i+j-1}t^{-\lambda_j^\vee+i}\right), \tag{2.8}$$

$$N_{\lambda\mu}(Q;q,t) = \prod_{(i,j)\in\lambda}\left(1-Q\,q^{\lambda_i-j}t^{\mu_j^\vee-i+1}\right)\prod_{(i,j)\in\mu}\left(1-Q\,q^{-\mu_i+j-1}t^{-\lambda_j^\vee+i}\right), \tag{2.9}$$

$$N_{\lambda\mu}(Q;q,t) = \Pi_0\left(-v^{-1}Q\,q^{\lambda}t^{\rho},\ t^{\mu^\vee}q^{\rho}\right)/\Pi_0\left(-v^{-1}Q\,t^{\rho},\ q^{\rho}\right), \tag{2.10}$$

$$N_{\lambda\mu}(Q;q,t) = \Pi_0\left(-v^{-1}Q\,t^{-\lambda^\vee}q^{-\rho},\ q^{-\mu}t^{-\rho}\right)/\Pi_0\left(-v^{-1}Q\,q^{-\rho},\ t^{-\rho}\right), \tag{2.11}$$

$$N_{\lambda\mu}(Q;q,t) = \Pi\left(Q\,q^{\lambda}t^{\rho},q^{-\mu}t^{-\rho};q,t\right)/\Pi\left(Q\,t^{\rho},t^{-\rho};q,t\right), \tag{2.12}$$

$$N_{\lambda\mu}(Q;q,t) = \Pi\left(Q\,t^{\mu^\vee}q^{\rho},t^{-\lambda^\vee}q^{-\rho};t^{-1},q^{-1}\right)/\Pi\left(Q\,q^{\rho},q^{-\rho};t^{-1},q^{-1}\right). \tag{2.13}$$

where

$$\Pi_0(-x,y) := \exp\left\{-\sum_{n>0}\frac{1}{n}p_n(x)p_n(y)\right\} = \prod_{i,j}(1-x_iy_j), \tag{2.14}$$

$$\Pi(vx,y;q,t) := \exp\left\{\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}p_n(x)p_n(y)\right\} = \begin{cases}\prod_{i,j}\dfrac{(ux_iy_j;q)_\infty}{(vx_iy_j;q)_\infty}, & |q|<1\\ \prod_{i,j}\dfrac{(u^{-1}x_iy_j;q^{-1})_\infty}{(v^{-1}x_iy_j;q^{-1})_\infty}, & |q^{-1}|<1.\end{cases} \tag{2.15}$$

Here $(x;q)_\infty$ is the q-shifted factorial $(x;q)_\infty := \prod_{k\in\mathbb{Z}_{\geq 0}}(1-q^kx)$. Note that when $|q^{-1}|,|t^{-1}|<1$, (2.10) is written as

$$\prod_{i,j=1}^{\infty}\left(1-Q\,q^{\lambda_i-j}t^{\mu_j^\vee-i+1}\right)/\left(1-Q\,t^{1-i}q^{-j}\right). \tag{2.16}$$

[6] In terms of $||\lambda||^2 := \sum_i \lambda_i^2 = 2\sum_{s\in\lambda}(a(s)+\frac{1}{2})$, this is $f_\lambda(q,t) = (-1)^{|\lambda|}q^{\frac{1}{2}||\lambda||^2}t^{-\frac{1}{2}||\lambda^\vee||^2}$.

and also when $|q| < 1$, (2.12) is

$$\prod_{i,j=1}^{\infty} \frac{\left(Q\, q^{\lambda_i-\mu_j} t^{j-i+1}; q\right)_\infty}{\left(Q\, q^{\lambda_i-\mu_j} t^{j-i}; q\right)_\infty} \frac{\left(Q\, t^{j-i}; q\right)_\infty}{\left(Q\, t^{j-i+1}; q\right)_\infty}. \tag{2.17}$$

The equivalence of these six expressions are proved in app. A. The first formula, (2.8), is given by [6], and (2.10)–(2.12) by [31]. In this article, we mainly use (2.9) and (2.10).

2.2 Another form of the partition function

Let us transform Nekrasov's partition function Z_m^{inst} so that it becomes transparent, and compare it with the amplitude constructed by the method of topological vertex. Using (E.9), i.e.

$$\prod_{(i,j)\in\lambda} q^{\mu_i-j} \prod_{(i,j)\in\mu} q^{-\lambda_i+j-1} = \prod_{(i,j)\in\mu} q^{\mu_i-j} \prod_{(i,j)\in\lambda} q^{-\lambda_i+j-1}, \tag{2.18}$$

we can show, from (2.8), that

$$\begin{aligned} N_{\mu\lambda}\left(Q^{-1}; q, t\right) &= N_{\lambda\mu}\left(v^2 Q; q, t\right) Q^{-|\lambda|-|\mu|} f_\mu(q,t) / f_\lambda(q,t) \\ &= N_{\mu^\vee\lambda^\vee}(Q; t, q)\, (vQ)^{-|\lambda|-|\mu|} f_\mu(q,t) / f_\lambda(q,t)\ . \end{aligned} \tag{2.19}$$

Here we use (2.33). Hence we have

$$\begin{aligned} \prod_{\alpha<\beta}^{N_c} N_{\lambda_\beta\lambda_\alpha}\left(Q_{\beta,\alpha}; q, t\right) &= \prod_{\alpha<\beta}^{N_c} N_{\lambda_\beta{}^\vee\lambda_\alpha{}^\vee}\left(Q_{\alpha,\beta}; t, q\right) \\ &\times \prod_{\alpha=1}^{N_c} \left(v^{N_c-1} \prod_{\beta=1}^{\alpha-1} Q_{\beta,\beta+1}^{\beta} \prod_{\beta=\alpha}^{N_c-1} Q_{\beta,\beta+1}^{N_c-\beta} \right)^{-|\lambda_\alpha|} f_{\lambda_\alpha}(q,t)^{-N_c+2\alpha-1}, \end{aligned} \tag{2.20}$$

and thus Nekrasov's formula (2.4) is rewritten as

$$Z_m^{\text{inst}} = \sum_{\lambda_1,\cdots,\lambda_{N_c}} \frac{\prod_{\alpha=1}^{N_c} \Lambda_{\alpha,m}{}^{|\lambda_\alpha|} f_{\lambda_\alpha}(q,t)^{N_c-m-2\alpha+1}}{\prod_{\alpha<\beta}^{N_c} N_{\lambda_\alpha\lambda_\beta}\left(Q_{\alpha,\beta}; q, t\right) N_{\lambda_\beta{}^\vee\lambda_\alpha{}^\vee}\left(Q_{\alpha,\beta}; t, q\right) \prod_{\alpha=1}^{N_c} N_{\lambda_\alpha\lambda_\alpha}(1; q, t)}, \tag{2.21}$$

where

$$\Lambda_{\alpha,m} := v^{-1}\Lambda^{2N_c} \left(-\mathbf{e}_\alpha\right)^{-m} \prod_{\beta=1}^{\alpha-1} Q_{\beta,\beta+1}^{\beta} \prod_{\beta=\alpha}^{N_c-1} Q_{\beta,\beta+1}^{N_c-\beta}. \tag{2.22}$$

For example, for $SU(2)$ theory we have

$$\begin{aligned} Z_m^{\text{inst}} &= \sum_{\lambda_1,\lambda_2} \frac{\left(v^{-1}\Lambda^4 Q_H\right)^{|\lambda|} (-\mathbf{e}_1)^{-m|\lambda_1|} (-\mathbf{e}_2)^{-m|\lambda_2|} f_{\lambda_1}(q,t)^{1-m} f_{\lambda_2}(q,t)^{-1-m}}{N_{\lambda_1\lambda_2}(Q_H; q, t)\, N_{\lambda_2{}^\vee\lambda_1{}^\vee}(Q_H; t, q)\, N_{\lambda_1\lambda_1}(1; q, t)\, N_{\lambda_2\lambda_2}(1; q, t)} \\ &= \sum_{\lambda_1,\lambda_2} \frac{\left(v^{-1}\Lambda^4 Q_H\right)^{|\lambda|} (-\mathbf{e}_1)^{-m|\lambda_1|} (-\mathbf{e}_2)^{-m|\lambda_2|} f_{\lambda_1}(q,t)^{1-m} f_{\lambda_2}(q,t)^{-1-m}}{N_{\lambda_1\lambda_1}(1; q, t)\, N_{\lambda_2\lambda_2}(1; q, t)} \end{aligned}$$

$$\times \frac{\Pi_0\left(-v^{-1}Q_H t^{\rho},\ q^{\rho}\right)\Pi_0\left(-vQ_H q^{\rho},\ t^{\rho}\right)}{\Pi_0\left(-v^{-1}Q_H q^{\lambda_1}t^{\rho},\ t^{\lambda_2^\vee}q^{\rho}\right)\Pi_0\left(-vQ_H t^{\lambda_2^\vee}q^{\rho},\ q^{\lambda_1}t^{\rho}\right)}\ , \tag{2.23}$$

where

$$N_{\lambda\lambda}\left(1;q,t\right) = \prod_{s\in\lambda}\left(1-q^{a(s)}t^{\ell(s)+1}\right)\left(1-q^{-a(s)-1}t^{-\ell(s)}\right), \tag{2.24}$$

and $Q_H = Q_{12}$. It is this form of Nekrasov's partition function that is obtained from the refined topological vertex with formulas of the Macdonald functions.

2.3 Symmetry as a character of $Spin(4)$

If Nekrasov's partition function gives the generating function of refined BPS state counting in the compactification of M theory on local Calabi-Yau spaces, it has to be a character of $Spin(4) \simeq SU(2)_L \times SU(2)_R$, since the spin of massive BPS particle in five dimensions is a representation of $Spin(4)$. In general, if a function $f(u,v)$ in two variables (u,v) is invariant under both $u \to u^{-1}$ and $v \to v^{-1}$, it is a linear combination of $Spin(4)$ characters

$$f(t,q) = \sum_{(s_L,s_R)} a_{(s_L,s_R)}\chi_{s_L}(u)\chi_{s_R}(v)\ , \tag{2.25}$$

where

$$\chi_n(z) := z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n} = \frac{z^{n+1}-z^{-n-1}}{z-z^{-1}}\ , \tag{2.26}$$

is the character of the irreducible representation of $SU(2)$ with spin $n/2$. Hence, if the k-instanton part $Z^{(k)}(q,t)$ of the partition function is invariant under the transformations $r_L : (q,t) \to (q^{-1},t^{-1})$ and $r_R : (q,t) \to (t,q)$. $Z^{(k)}(q,t)$ is expanded as

$$Z^{(k)}(q,t) = \sum_{(s_L,s_R)} a^{(k)}_{(s_L,s_R)}\chi_{s_L}(u)\chi_{s_R}(v)\ , \tag{2.27}$$

with rational coefficients $a^{(k)}_{(s_L,s_R)}$. Recall that $u = \sqrt{qt}$ and $v = \sqrt{q/t}$. Actually we will find an appropriate scaling of $Z^{(k)}(q,t)$ depending on the instanton number k is necessary for the genuine invariance under the above transformations. If the partition function takes the form (2.27), then the k-instanton part $F^{(k)}(q,t)$ of the free energy is also a linear combination of $Spin(4)$ characters. Furthermore, if the pole structure of the free energy is appropriate, we can factor out the character of the half-hypermultiplet and subtract the multicovering contributions to obtain the expansion of the total free energy

in the Gopakumar-Vafa form:

$$
\begin{aligned}
F = \log Z &= \sum_{k=0}^{\infty} F^{(k)}(Q_\beta; q, t) \\
&= \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{(j_L, j_R)} \sum_{n=1}^{\infty} \frac{N_\beta^{(j_L, j_R)} u^n v^n}{n(u^n v^n - 1)(u^n - v^n)} \chi_{n \cdot j_L}(u) \chi_{n \cdot j_R}(v) Q_\beta^n \ .
\end{aligned} \tag{2.28}
$$

The coefficients $N_\beta^{(j_L, j_R)}$ of the expansion (2.28) are conjectured to be nonnegative integers, since from the viewpoint of the Calabi-Yau compactification of M theory they are interpreted as multiplicities of the five-dimensional BPS particles arising from $M2$ branes wrapping on a two-cycle $\beta \in H_2(X, \mathbb{Z})$ in the Calabi-Yau 3-fold X. We have checked the integrality of the refined BPS state counting from the $SU(2)$ and $SU(3)$ partition functions up to instanton number 2. The result is presented in appendix C.

Since the transformation $(q, t) \to (t^{-1}, q^{-1})$ is compensated by the transpose of colored partitions, we have

$$
\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; t^{-1}, q^{-1}\right) = \prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha^\vee \lambda_\beta^\vee} \left(Q_{\alpha,\beta}; q, t\right) \ . \tag{2.29}
$$

By (2.19), we also find that

$$
\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}^{-1}; q^{-1}, t^{-1}\right) = \left(\frac{q}{t}\right)^{N_c|\lambda|} \prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; q, t\right) \ . \tag{2.30}
$$

Therefore, we obtain

$$
\begin{aligned}
\sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; t^{-1}, q^{-1}\right)} &= \sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; q, t\right)} \ , \\
\sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}^{-1}; q^{-1}, t^{-1}\right)} &= \left(\frac{t}{q}\right)^{N_c|\lambda|} \sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; q, t\right)} \ .
\end{aligned} \tag{2.31}
$$

Thus if we can prove

$$
\sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}^{-1}; q, t\right)} = \sum_{\{\lambda_\alpha\}, |\lambda|=k} \frac{1}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda_\alpha \lambda_\beta} \left(Q_{\alpha,\beta}; q, t\right)} \ , \tag{2.32}
$$

the partition function $Z_m^{\rm inst}$ is invariant under both of the reflections r_L and r_R. It is easy to see that the property (2.32) is valid for $N_c = 2$, since the exchange of two partitions

effectively induces $\mathbf{e}_\alpha \to \mathbf{e}_\alpha^{-1}$. For $N_c > 2$ the validity of (2.32) seems nontrivial. The overall reflection of the roots $\mathbf{e}_\alpha \to \mathbf{e}_\alpha^{-1}$ cannot be induced by any permutation of colored partitions. However, we have checked by explicit computations that (2.32) is true for $N_c = 3$ and $k = 1, 2$. This is consistent with the computation in appendix C, where we obtain the results of refined BPS state counting from Nekrasov's partition function of $SU(3)$ gauge theory.

The symmetry of Nekrasov's partition function with the Chern-Simons coupling can also be derived easily. Because of (A.20), the factor $N_{\lambda\mu}(Q; q, t)$ enjoys the following duality relations:

$$N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}\left(v^{-1}Q; q^{-1}, t^{-1}\right) = N_{\mu^\vee\lambda^\vee}\left(v^{-1}Q; t, q\right). \tag{2.33}$$

From the expression (2.9), we also have

$$N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}\left(vQ^{-1}; q, t\right) Q^{|\lambda|+|\mu|} f_\lambda(q, t) / f_\mu(q, t). \tag{2.34}$$

Since

$$N_{\lambda\mu}\left(v^2Q; q, t\right) N_{\mu\lambda}\left(v^2Q^{-1}; q, t\right) = N_{\lambda\mu}(Q; q, t)\, N_{\mu\lambda}\left(Q^{-1}; q, t\right) v^{2(|\lambda|+|\mu|)}, \tag{2.35}$$

we find that

$$\begin{aligned} N_{\lambda\mu}(Q; q, t)\, N_{\mu\lambda}\left(Q^{-1}; q, t\right) v^{|\lambda|+|\mu|} &= N_{\lambda\mu}\left(Q^{-1}; q^{-1}, t^{-1}\right) N_{\mu\lambda}\left(Q; q^{-1}, t^{-1}\right) v^{-|\lambda|-|\mu|} \\ &= N_{\lambda^\vee\mu^\vee}\left(Q^{-1}; t, q\right) N_{\mu^\vee\lambda^\vee}(Q; t, q)\, v^{-|\lambda|-|\mu|}. \end{aligned} \tag{2.36}$$

Thus, Nekrasov's partition function Z_m^{inst} has the following symmetries:

$$\begin{aligned} Z_m^{\text{inst}}(\mathbf{e}_1, \cdots, \mathbf{e}_{N_c}, \Lambda; q, t) &= Z_{-m}^{\text{inst}}\left(\mathbf{e}_1^{-1}, \cdots, \mathbf{e}_{N_c}^{-1}, \Lambda; q^{-1}, t^{-1}\right) \\ &= Z_{-m}^{\text{inst}}\left(\mathbf{e}_1^{-1}, \cdots, \mathbf{e}_{N_c}^{-1}, \Lambda; t, q\right). \end{aligned} \tag{2.37}$$

3 Geometric Engineering and Toric Geometry

In this section following [4, 5, 8, 11], we review the toric geometry that is necessary for geometric engineering. Geometric engineering tells how to obtain $\mathcal{N} = 2$ $SU(N_c)$ super Yang-Mills theory with N_f fundamental matters from type II(A) string theory on local Calabi-Yau manifold K_S, the canonical bundle of a 4-cycle S. The (toric) geometry of the 4-cycle S can be described by the (dual) toric diagram. The prescription of the geometric engineering implies that the toric diagram of S has N_c horizontal internal edges

("color" $D5$ branes) and N_f horizontal external edges ("flavor" $D5$ branes). For example, the vertical distance of "color" $D5$ branes represents vacuum expectation values of the Higgs fields or the mass of W bosons. The matter fermions are given by fundamental strings connecting a "color" $D5$ brane and a "flavor" $D5$ brane. The vertical distance of a "color" $D5$ brane and a "flavor" $D5$ brane represents the mass of the corresponding matter fermion.

One of the properties of toric diagrams that arise from geometric engineering is that each vertex has a unique horizontal edge. In the following we will consider toric diagrams in which we specify the horizontal edges as distinguished. In the computation by the method of topological vertex, we cut the internal horizontal edges. Then the contribution of each component is given by an amplitude of "the vertex on a strip"[32]. By gluing these amplitudes we obtain the partition function for the local toric Calabi-Yau manifold K_S.

In the compactification of type IIA string theory on local Calabi-Yau manifold, $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theory is geometrically engineered by ALE fibration of A_{N-1} type over the rational curve $\mathbf{P}^1$. The fiber consists of a chain of $N-1$ rational curves whose intersection form is given by the minus of the Cartan matrix of A_{N-1}. The holomorphic 2-cycles in the fiber are in one-to-one correspondence with the positive roots of A_{N-1}. The (dual) toric diagram takes the form of "ladder" diagram with N parallel horizontal edges. In the toric diagram the faces correspond to compact 4-cycles (divisors).

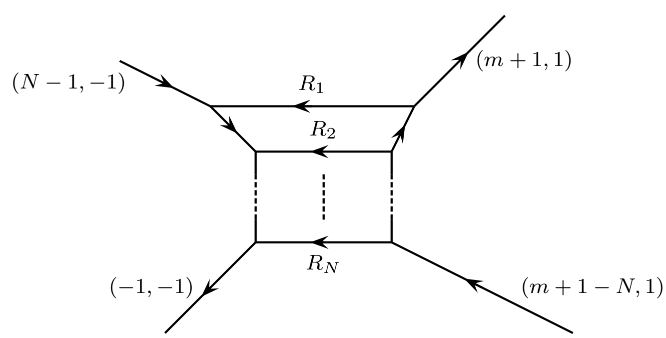


Figure 1: Ladder diagram for $SU(N)$ gauge theory. There are $N+1$ possible toric diagrams ($m=0,\cdots N$).

In the ladder diagram of ALE fibration over $\mathbf{P}^1$, we find $N-1$ divisors, all of which are $\mathbf{P}^1$ fibration over $\mathbf{P}^1$, namely the Hirzebruch surfaces. The degree of the Hirzebruch surface can be determined by the (relative) slopes of the vertical edges of the face. We will denote the Hirzebruch surface of degree n by $\mathbf{F}_n$. It is known that for each N there

are $N+1$ types of such geometry, which we label by $m = 0, 1, \cdots N$ [8, 11]. The integer m is related to the coupling constant of five-dimensional Chern-Simons coupling [30]. Let us call such geometry toric $SU(N)_m$ geometry. We can characterize the toric $SU(N)_m$ geometry by saying that its compact 4-cycles are $\{\mathbf{F}_{N-2+m}, \mathbf{F}_{N-4+m}, \cdots \mathbf{F}_{-N+2+m}\}$.

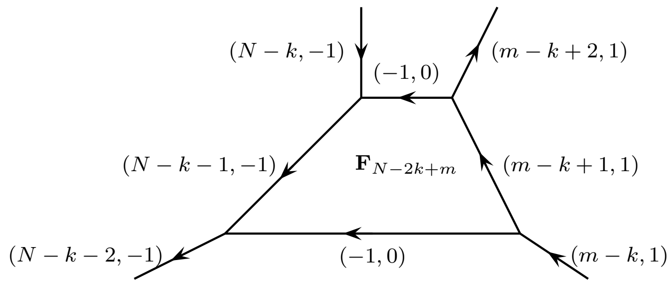


Figure 2: Subdiagram of the k-th divisor of $SU(N)_m$ geometry $(1 \le k \le N-1)$

The Kähler parameters of $SU(N)_m$ geometry are T_B of the base space $\mathbf{P}^1$ and T_{F_i} $(i = 1, \cdots, N-1)$ of the fiber which is a chain of $(N-1)$ $\mathbf{P}^1$'s. In the subdiagram of Figure 2, the rational curves of both side edges correspond to the fiber of $\mathbf{F}_{N-2k+2}$, and their Kähler parameters are T_{F_k}. On the other hand, if we denote the Kähler parameters of the upper and the lower edges by T_{B_k} and $T_{B_{k+1}}$, respectively. The difference is related to the degree of the Hirzebruch surface as follows:

$$T_{B_k} - T_{B_{k+1}} = (N - 2k + m) T_{F_k} \ . \tag{3.1}$$

From the recursion relation (3.1) we find, if $N + m = 2r + 1$ is odd, that

$$\begin{aligned} T_{B_r} &:= T_B \ , \\ T_{B_i} &= T_B + \sum_{j=i}^{r} (N + m - 2j) T_{F_j} \ , \quad (1 \le i \le r-1) \ , \\ T_{B_i} &= T_B + \sum_{j=r+1}^{i-1} (2j - N - m) T_{F_j} \ , \quad (r+1 \le i \le N) \ , \end{aligned} \tag{3.2}$$

and if $N + m = 2r$ is even,

$$\begin{aligned} T_{B_r} &= T_{B_{r+1}} := T_B \ , \\ T_{B_i} &= T_B + \sum_{j=i}^{r-1} (N + m - 2j) T_{F_j} \ , \quad (1 \le i \le r-1) \ , \\ T_{B_i} &= T_B + \sum_{j=r+1}^{i-1} (2j - N - m) T_{F_j} \ , \quad (r+2 \le i \le N) \ . \end{aligned} \tag{3.3}$$

In (3.2) and (3.3) we take the first relations as initial conditions in solving (3.1).

From the slope of each edge in Figure 1, we can also compute the framing index by the rule to be explained in section 4.2. Let us denote the index of left, right, upper and lower edges by $n_{L,k}, n_{R,k}, n_{B,k}$ and $n_{B,k+1}$, respectively. Then we compute

$$\begin{aligned} n_{B,k} &= (m-k+1,1)\wedge(-N+k,1) = N+m-2k+1, \\ n_{L,k} &= (-1,0)\wedge(N-k-2,-1) = 1, \\ n_{R,k} &= (m-k,1)\wedge(-1,0) = 1. \end{aligned} \tag{3.4}$$

Note that $n_{L,k}$ and $n_{R,k}$ are independent of k. By definition the framing index changes the sign, if we reverse the orientation of the edge, or replace the representation associated to the edge by its transpose. We will use these framing indices in the computation of the partition function by gluing the refined topological vertices.

4 Refined Topological Vertex

In [14], we defined the refined topological vertex which is written not by the Schur functions but by the Macdonald functions. Here we slightly modify it by improving the framing factor.

4.1 Refined topological vertex

Let $P_{\lambda/\mu}(x;q,t)$ and $\langle P_\lambda|P_\lambda\rangle_{q,t}$ be the Macdonald function in the infinite number of variables $x=(x_1,x_2,\cdots)$ and its scalar product, respectively, defined in Appendix B. We introduce an involution ι acting on the power sum function $p_n(x)$ by $\iota(p_n)=-p_n$. For example,

$$\iota p_n(q^\lambda t^\rho) = -\sum_{i=1}^{\infty}(q^{n\lambda_i}-1)t^{n(\frac{1}{2}-i)} - \frac{1}{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}. \tag{4.1}$$

Note that $\iota p_n(t^\rho) = -p_n(t^\rho) = p_n(t^{-\rho})$.

We define a vertex $V_{\mu\lambda}{}^{\nu}$ as follows:[7]

$$V_{\mu\lambda}{}^{\nu} := P_\lambda(t^\rho;q,t)\sum_\sigma \iota P_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ P_{\nu/\sigma}(q^\lambda t^\rho;q,t)v^{|\sigma|}, \tag{4.2}$$

[7] Although we will show that the Nekrasov formula is represented by our vertex $V_{\mu\lambda}{}^{\nu}$, one can also produce it through the following vertex without the involution ι

$$U_{\mu\lambda}{}^{\nu} = P_\lambda(t^\rho;q,t)\sum_\sigma P_{\mu/\sigma}(q^{-\lambda}t^{-\rho};q^{-1},t)\ P_{\nu/\sigma}(q^\lambda t^\rho;q^{-1},t)\langle P_\sigma|P_\sigma\rangle_{q,t}.$$

where

$$P_\lambda(t^\rho;q,t)=\prod_{s\in\lambda}\frac{(-1)t^{\frac12}q^{a(s)}}{1-q^{a(s)}t^{\ell(s)+1}},\qquad P_{\lambda^\vee}(-q^\rho;t,q)=\prod_{s\in\lambda}\frac{(-1)q^{-\frac12}q^{-a(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}},\tag{4.3}$$

which follows by substituting $Q=0$ into (B.21). From (B.20), ${V_{\mu\lambda}}^{\nu}$ is rewritten as

$${V_{\mu\lambda}}^{\nu}=P_\lambda(t^\rho;q,t)\sum_\sigma \iota P_{\mu/\sigma}(q^{-\lambda}t^{-\rho};q,t)\ P_{\nu/\sigma}(q^\lambda t^\rho;q,t)\langle P_\sigma|P_\sigma\rangle_{q,t}\ g_\mu(q,t),\tag{4.4}$$

with

$$g_\lambda(q,t):=\frac{v^{|\lambda|}}{\langle P_\lambda|P_\lambda\rangle_{q,t}}=\prod_{s\in\lambda}\left(\frac{q}{t}\right)^{\frac12}\frac{1-q^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)+1}t^{\ell(s)}},\tag{4.5}$$

which satisfies

$$g_\lambda(q,t)=g_\lambda(q^{-1},t^{-1})=g_{\lambda^\vee}(t,q)^{-1}.\tag{4.6}$$

From (4.4), (B.28) and (B.27), one can show the symmetry[8]

$$g_\lambda(q,t)^{-1}{V_{\lambda\bullet}}^{\bullet}={V_{\bullet\lambda}}^{\bullet}={V_{\bullet\bullet}}^{\lambda},\tag{4.7}$$

$$g_\mu(q,t)^{-1}{V_{\mu\bullet}}^{\nu}=g_\nu(q,t)^{-1}{V_{\nu\bullet}}^{\mu},\tag{4.8}$$

$${V_{\bullet\lambda}}^{\nu}={V_{\bullet\nu}}^{\lambda}.\tag{4.9}$$

Incorporating the framing factor, we define our refined topological vertices ${C_{\mu\lambda}}^{\nu}(q,t)$ and ${C^{\mu\lambda}}_{\nu}(q,t)$ as follows:

$$\begin{aligned}{C_{\mu\lambda}}^{\nu}(q,t)&:={V_{\mu\lambda}}^{\nu}v^{-|\nu|}f_\nu(q,t)^{-1}\qquad\qquad(4.10)\\&=P_\lambda(t^\rho;q,t)\sum_\sigma\iota P_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ P_{\nu/\sigma}(q^\lambda t^\rho;q,t)v^{|\sigma|-|\nu|}f_\nu(q,t)^{-1},\\{C^{\mu\lambda}}_{\nu}(q,t)&:={C_{\mu^\vee\lambda^\vee}}^{\nu^\vee}(t,q)(-1)^{|\lambda|+|\mu|+|\nu|}\qquad\qquad(4.11)\\&=P_{\lambda^\vee}(-q^\rho;t,q)\sum_\sigma P_{\nu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ \iota P_{\mu/\sigma}(q^\lambda t^\rho;q,t)v^{-|\sigma|+|\nu|}f_\nu(q,t).\end{aligned}$$

The lower and the upper indices correspond to the incoming and the outgoing representations, respectively, and the edges of the topological vertex are ordered clockwise. Although only the refined vertices of the above types are mainly used in this article, the following vertices may also be useful:

$$\begin{aligned}{C^{\mu}}_{\lambda\nu}(q,t)&:={C_{\mu\lambda}}^{\nu}(q,t)v^{|\mu|+|\nu|}f_\mu(q,t)\,f_\nu(q,t)\qquad\qquad(4.12)\\&=P_\lambda(t^\rho;q,t)\sum_\sigma\iota P_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ P_{\nu/\sigma}(q^\lambda t^\rho;q,t)v^{|\sigma|+|\mu|}f_\mu(q,t),\end{aligned}$$

[8] If we replace Macdonald functions $P_{\lambda/\mu}(x;q,t)$'s in ${V_{\mu\lambda}}^{\nu}$ by "normalized" Macdonald functions $\widetilde{P}_{\lambda/\mu}(x;q,t):=P_{\lambda/\mu}(x;q,t)\sqrt{g_\lambda(q,t)/g_\mu(q,t)}$, then the g factors in (4.7)–(4.9) disappear. Because $\widetilde{P}_\lambda(x;q,t)\widetilde{P}_{\lambda^\vee}(y;t,q)=P_\lambda(x;q,t)P_{\lambda^\vee}(y;t,q)$, all results in this article remain the same even if we use the normalized Macdonald functions $\widetilde{P}_{\lambda/\mu}(x;q,t)$.

$$
\begin{aligned}
C_\mu{}^{\lambda\nu}(q,t) &:= C^{\mu\lambda}{}_\nu(q,t) v^{-|\mu|-|\nu|} f_\mu\,(q,t)^{-1}\, f_\nu\,(q,t)^{-1} = C^{\mu^\vee}{}_{\lambda^\vee\nu^\vee}(t,q)(-1)^{|\lambda|+|\mu|+|\nu|} \\
&= P_{\lambda^\vee}(-q^\rho;t,q) \sum_\sigma P_{\nu^\vee/\sigma^\vee}(-t^{\lambda^\vee} q^\rho;t,q)\ \iota P_{\mu/\sigma}(q^\lambda t^\rho;q,t) v^{-|\sigma|-|\mu|} f_\mu\,(q,t)^{-1}\,.
\end{aligned}
\tag{4.13}
$$

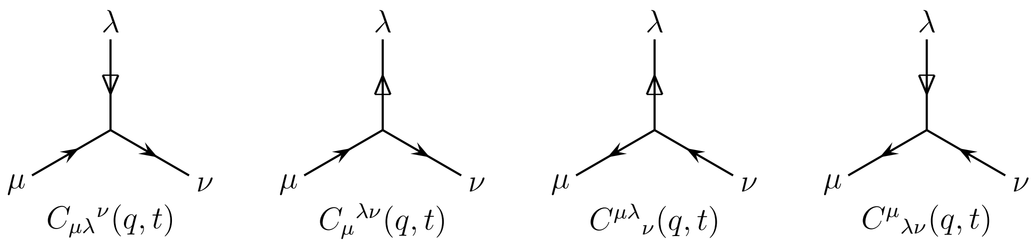


Figure 3: Refined topological vertex: the represantation for the preferred direction, i.e. the middle index λ, is indicated by the white arrow.

Note that, when $q = t$, the topological vertex in [3] is

$$
C_{\mu\lambda\nu}(q) = s_\lambda(q^\rho) \sum_\sigma s_{\mu/\sigma}(q^{\lambda^\vee+\rho})\ s_{\nu^\vee/\sigma}(q^{\lambda+\rho}) \prod_{s\in\nu} q^{a(s)-\ell(s)}. \tag{4.14}
$$

Since $s_{\mu/\sigma}(q^{\lambda^\vee+\rho}) = \iota s_{\mu/\sigma}(q^{-\lambda-\rho})$, which follows from (A.11), our refined topological vertex $\lim_{t\to q} C_{\mu\lambda}{}^\nu(q,t)$ coincides with the topological vertex $C_{\mu\lambda\nu^\vee}(q)$. It is well-known that in the operator formalism the Schur functions are realized in terms of free fermions. Although we have no fermionic realization of the Macdonald functions, they are described by bosons as shown in [33]. Our refined topological vertex has a bosonic realization by using that for the Macdonald functions.

4.2 Gluing rules

Here we show our gluing rules for constructing the partition function from a web diagram. Let us consider a graph with trivalent vertices and edges. Each edge is associated with an integer vector $\boldsymbol{v} = (v_1, v_2) \in \mathbb{Z}^2$. Hence the trivalent vertex with edges indexed by (i,j,k) in the counterclockwise ordering is associated with a triplet of integer vectors $(\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k)$. If we choose these vectors to be outgoing, they should satisfy the following conditions

$$
\boldsymbol{v}_i + \boldsymbol{v}_j + \boldsymbol{v}_k = \boldsymbol{0}, \qquad \boldsymbol{v}_i \wedge \boldsymbol{v}_j = 1, \qquad (\boldsymbol{v}_j \wedge \boldsymbol{v}_k = \boldsymbol{v}_k \wedge \boldsymbol{v}_i = 1), \tag{4.15}
$$

with $\boldsymbol{v}_i \wedge \boldsymbol{v}_j := v_{i,1}v_{j,2} - v_{i,2}v_{j,1}$. These correspond to the Calabi-Yau condition and the smoothness condition. Since the refined topological vertex has no cyclic symmetry,

we should specify a preferred direction. Therefore one of these three vectors should be the preferred one and we denote it by white arrow. Note that if we choose the middle edge as the preferred direction; $\boldsymbol{v}_j = (-1, 0)$, then the condition (4.15) implies that $\boldsymbol{v}_i = (a, 1), \boldsymbol{v}_k = (b, -1)$ with $a + b = 1$.

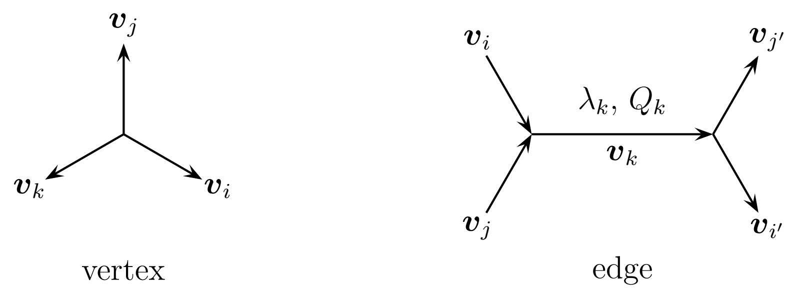


Figure 4: Gluing rules

Let $(\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k)$ and $(\boldsymbol{v}_k, \boldsymbol{v}_{i'}, \boldsymbol{v}_{j'})$ be the vectors associated with the vertices at the origin and at the end of the vector $\boldsymbol{v}_k$ of the kth edge, respectively. If we choose so that $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are incoming and $\boldsymbol{v}_k$ and $\boldsymbol{v}_{i'}$ are outgoing, then the framing index n_k of the kth edge is defined by

$$n_k := \boldsymbol{v}_i \wedge \boldsymbol{v}_{i'} = \boldsymbol{v}_j \wedge \boldsymbol{v}_{j'}. \tag{4.16}$$

Each edge is associated also with a Young diagram λ and a Kähler parameter $Q \in \mathbb{C}$ so that the propagator for the kth edge is defined as

$$Q_k^{|\lambda_k|} f_{\lambda_k}(q, t)^{n_k}, \tag{4.17}$$

and we glue the amplitudes by summing over the representation λ on each edge.

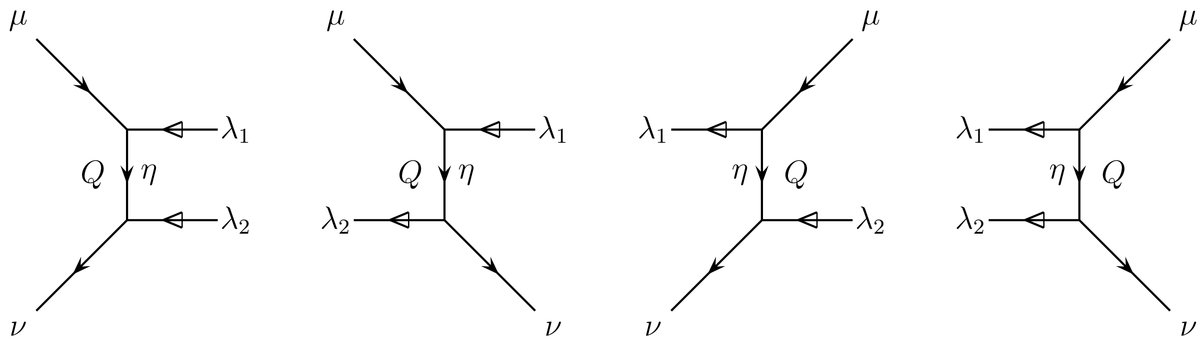
5 Four-Point Functions

Here we show how to calculate the partition functions. The building blocks for them are the following four-point functions.

5.1 Building blocks

Assume that each vertex has a horizontal edge, which we take as the preferred direction. Fix the orientation of the preferred direction, say $(-1, 0)$; then we have four possibilities of the configuration of two horizontal edges [Fig. 5]. Although the slopes and directions

of “vertical,” i.e. nonhorizontal, edges can be arbitrary, we show in Figure 5 the simplest one whose internal edge is orthogonal to the preferred direction and we tentatively take the orientation of “vertical” edges from the top to the bottom. The framing index is 1, 0, 0 and -1, respectively. They are independent of the slope of “vertical” edges, but change the sign according to the orientations.



$Z_{\mu\lambda_1\lambda_2}{}^{\nu}(Q;q,t)$ $\quad$ $Z_{\mu\lambda^1}{}^{\lambda^2\nu}(Q;q,t)$ $\quad$ $Z_{\mu}{}^{\lambda^1}{}_{\lambda^2}{}^{\nu}(Q;q,t)$ $\quad$ $Z_{\mu}{}^{\lambda_1\lambda_2\nu}(Q;q,t)$

Figure 5: Four-point function: the framing indices of the internal lines are 1, 0, 0 and -1 from the left.

We order the three edges at each vertex in the clockwise direction such that the preferred direction is in the middle position. This fixes the ordering of three edges uniquely. The lower and upper indices of the refined vertex correspond to the incoming and the outgoing representation. Then the following four-point functions are building blocks for the partition function:

$$
\begin{aligned}
Z_{\mu\lambda_1\lambda_2}{}^{\nu}(Q;q,t) &:= \sum_{\eta} C_{\mu\lambda_1}{}^{\eta}(q,t)C_{\eta\lambda_2}{}^{\nu}(q,t)Q^{|\eta|}f_{\eta}(q,t)\,,\\
Z_{\mu\lambda_1}{}^{\lambda_2\nu}(Q;q,t) &:= \sum_{\eta} C_{\mu\lambda_1}{}^{\eta}(q,t)C^{\nu\lambda_2}{}_{\eta}(q,t)Q^{|\eta|},\\
Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t) &:= \sum_{\eta} C^{\eta\lambda_1}{}_{\mu}(q,t)C_{\eta\lambda_2}{}^{\nu}(q,t)Q^{|\eta|},\\
Z_{\mu}{}^{\lambda_1\lambda_2\nu}(Q;q,t) &:= \sum_{\eta} C^{\eta\lambda_1}{}_{\mu}(q,t)C^{\nu\lambda_2}{}_{\eta}(q,t)Q^{|\eta|}f_{\eta}(q,t)^{-1}\,.
\end{aligned}
\tag{5.1}
$$

Note that

$$
\begin{aligned}
Z_{\mu}{}^{\lambda_1\lambda_2\nu}(Q;q,t) &= \sum_{\eta^\vee} C_{\eta^\vee\lambda_1{}^\vee}{}^{\mu^\vee}(t,q)C_{\nu^\vee\lambda_2{}^\vee}{}^{\eta^\vee}(t,q)(-1)^{|\mu|+|\lambda_1|+|\lambda_2|+|\nu|}Q^{|\eta|}f_{\eta}(t,q)\\
&= Z_{\nu^\vee\lambda_2{}^\vee\lambda_1{}^\vee}{}^{\mu^\vee}(Q;t,q)(-1)^{|\mu|+|\lambda_1|+|\lambda_2|+|\nu|}.
\end{aligned}
\tag{5.2}
$$

We will show that $Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t)$ is related with $Z_{\mu\lambda_1}{}^{\lambda_2\nu}(Q;q,t)$ by the flop. If we take the orientation of "vertical" edges from the bottom to the top, the sign of the framing index changes. The corresponding four-point functions are written by $C^{\mu}{}_{\lambda\nu}(q,t)$'s and $C_{\mu}{}^{\lambda\nu}(q,t)$'s and they are the same as those in (5.1) up to the framing factors $f_{\mu}(q,t)^{\pm1} f_{\nu}(q,t)^{\pm1}$ for the outer "vertical" edges.

Although we have fixed a preferred direction in this article, we can change it in some special cases. Let

$$Z^{\bullet}{}_{\nu\bullet}{}^{\mu}(Q;q,t) := \sum_{\eta} C_{\bullet\eta}{}^{\mu}(q,t)C^{\bullet\eta}{}_{\nu}(q,t)Q^{|\eta|}, \tag{5.3}$$

then from (4.9), we have the following symmetry

$$Z_{\bullet\mu}{}^{\nu\bullet}(Q;q,t) = Z^{\bullet}{}_{\nu\bullet}{}^{\mu}(Q;q,t)v^{|\mu|-|\nu|}f_{\mu}(q,t)/f_{\nu}(q,t), \tag{5.4}$$

which changes the preferred direction.

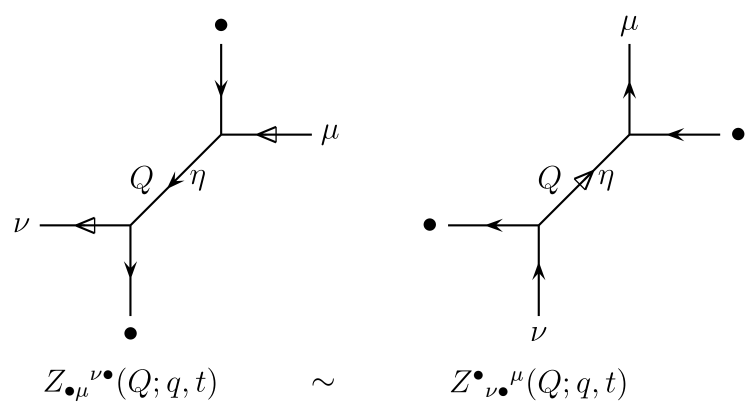


Figure 6: Changing the preferred direction

5.2 OPE formula

Next, we turn to showing some formulas for calculating the partition functions. Let us denote a symmetric function f in the set of variables $(x_1^1, x_2^1, \cdots, x_1^2, x_2^2, \cdots, x_1^N, x_2^N, \cdots)$ by $f\left(x^1, x^2, \cdots, x^N\right)$ or $f\left(\{x^i\}_{i=1}^N\right)$. To calculate the partition functions, the essential part is the following Cauchy formula for the Macdonald function,

$$\sum_{\lambda} P_{\lambda}(x;q,t)P_{\lambda^\vee}(y;t,q) = \Pi_0(x,y), \tag{5.5}$$

or, more generally,

$$\sum_{\lambda} P_{\lambda/\mu}(x;q,t)P_{\lambda^\vee/\nu^\vee}(y;t,q) = \Pi_0(x,y)\sum_{\lambda} P_{\mu^\vee/\lambda^\vee}(y;t,q)P_{\nu/\lambda}(x;q,t), \tag{5.6}$$

with $\Pi_0(x,y)$ in (2.14) and the adding formula

$$\sum_{\mu} P_{\lambda/\mu}(x;q,t)P_{\mu/\nu}(y;q,t) = P_{\lambda/\nu}(x,y;q,t). \tag{5.7}$$

Note that for $c \in \mathbb{C}$, $\Pi_0(cx,y) = \Pi_0(x,cy)$, and for our involution ι in (4.1), $\Pi_0(\iota x,y) = \Pi_0(x,\iota y) = \Pi_0(x,y)^{-1}$. Using these we have the following lemma.

Lemma. Let x, y, z and w be sets of variables and α, β and $\gamma \in \mathbb{C}$. Then

$$\begin{aligned}&\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}(x;t,q)\,P_{\eta/\sigma_1}(y;q,t)\,P_{\eta^\vee/\sigma_2{}^\vee}(z;t,q)\,P_{\nu/\sigma_2}(w;q,t)\,\alpha^{|\sigma_1|}\beta^{|\eta|}\gamma^{|\sigma_2|}\\ &=\sum_{\eta} P_{\mu^\vee/\eta^\vee}(x,\alpha\beta z;t,q)\,P_{\nu/\eta}(\beta\gamma y,w;q,t)\,(\alpha\beta\gamma)^{|\eta|}\,\Pi_0(y,\beta z)\,.\end{aligned} \tag{5.8}$$

Proof. Let $\alpha = a/b$, $\beta = b/c$, $\gamma = c/d$; then, from (B.11), (5.6) and (5.7), the left-hand side of the above equation is

$$\begin{aligned}&\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}\left(\frac{x}{a};t,q\right)P_{\eta/\sigma_1}(by;q,t)\,P_{\eta^\vee/\sigma_2{}^\vee}\left(\frac{z}{c};t,q\right)P_{\nu/\sigma_2}(dw;q,t)\,a^{|\mu|}d^{-|\nu|}\\ &=\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}\left(\frac{x}{a};t,q\right)P_{\sigma_1{}^\vee/\eta^\vee}\left(\frac{z}{c};t,q\right)P_{\sigma_2/\eta}(by;q,t)\,P_{\nu/\sigma_2}(dw;q,t)\,a^{|\mu|}d^{-|\nu|}\Pi_0\left(by,\frac{z}{c}\right)\\ &=\sum_{\eta} P_{\mu^\vee/\eta^\vee}\left(\frac{x}{a},\frac{z}{c};t,q\right)P_{\nu/\eta}(by,dw;q,t)\,a^{|\mu|}d^{-|\nu|}\Pi_0\left(by,\frac{z}{c}\right)\\ &=\sum_{\eta} P_{\mu^\vee/\eta^\vee}\left(x,\frac{a}{c}z;t,q\right)P_{\nu/\eta}\left(\frac{b}{d}y,w;q,t\right)\left(\frac{a}{d}\right)^{|\eta|}\Pi_0\left(y,\frac{b}{c}z\right),\end{aligned} \tag{5.9}$$

and the lemma is proven. □

Successively using this lemma, we obtain the following OPE formula, which is useful for calculating more general diagrams.

Proposition. Let x^i's be sets of variables, $c_{i,i+1} \in \mathbb{C}$ and $c_{i,j} := \prod_{k=i}^{j-1} c_{i,i+1}$. Then

$$\begin{aligned}&\sum_{\{\lambda_1,\lambda_2,\cdots,\lambda_{2N-1}\}}\prod_{i=1}^{N} P_{\lambda_{2i-2}{}^\vee/\lambda_{2i-1}{}^\vee}\left(x^{2i-1};t,q\right)P_{\lambda_{2i}/\lambda_{2i-1}}\left(x^{2i};q,t\right)\prod_{i=1}^{2N-1} c_{i,i+1}^{|\lambda_i|}\\ &=\sum_{\eta} P_{\lambda_0{}^\vee/\eta^\vee}\left(\{c_{1,2i-1}x^{2i-1}\}_{i=1}^{N};t,q\right)P_{\lambda_{2N}/\eta}\left(\{x^{2i}c_{2i,2N}\}_{i=1}^{N};q,t\right)c_{1,2N}^{|\eta|}\\ &\qquad\times\prod_{1\le i<j\le N}\Pi_0\left(x_{2i}\ ,\ c_{2i,2j-1}x_{2j-1}\right),\end{aligned} \tag{5.10}$$

for any integer $N \geq 2$.

Therefore the number of Young diagrams to perform summation reduces from $2N-1$ to one. If λ_0 or λ_{2N} is the trivial representation, then since $P_{\bullet/\lambda}(x;q,t) = \delta_{\bullet,\lambda}$, the number of Young diagrams for perform summation becomes zero. The trace over $\lambda_0 = \lambda_{2N}$ is also calculated by the trace formula explained in the next section. If we realize the Macdonald polynomials by bosons as in [33], these OPE formulas come from the operator product expansion of vertex operators.

5.3 Computations of four-point functions

Here we apply the OPE formula (5.8) to the above building blocks. Let x^α and y^α be the set of variables as $x^\alpha = q^{\lambda_\alpha} t^\rho$ and $y^\alpha = t^{\lambda_\alpha{}^\vee} q^\rho$, respectively. Then

$$\begin{aligned}
&Z_{\mu\lambda_1\lambda_2}{}^{\nu}(Q;q,t) = P_{\lambda_1}(t^\rho;q,t)\,P_{\lambda_2}(t^\rho;q,t)\,f_\nu(q,t)^{-1}\,v^{-|\nu|}\\
&\times\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}\left(-\iota y^1;t,q\right) P_{\eta/\sigma_1}\left(x^1;q,t\right) P_{\eta^\vee/\sigma_2{}^\vee}\left(-\iota y^2;t,q\right) P_{\nu/\sigma_2}\left(x^2;q,t\right) v^{|\sigma_1|+|\sigma_2|}\left(v^{-1}Q\right)^{|\eta|},\\
&Z_{\mu\lambda_1}{}^{\lambda_2\nu}(Q;q,t) = P_{\lambda_1}(t^\rho;q,t)\,P_{\lambda_2{}^\vee}(-q^\rho;t,q)\\
&\times\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}\left(-\iota y^1;t,q\right) P_{\eta/\sigma_1}\left(x^1;q,t\right) P_{\eta^\vee/\sigma_2{}^\vee}\left(-y^2;t,q\right) P_{\nu/\sigma_2}\left(\iota x^2;q,t\right) v^{|\sigma_1|-|\sigma_2|}Q^{|\eta|},\\
&Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t) = P_{\lambda_1{}^\vee}(-q^\rho;t,q)\,P_{\lambda_2}(t^\rho;q,t)\,f_\mu(q,t)\,f_\nu(q,t)^{-1}\,v^{|\mu|-|\nu|}\\
&\times\sum_{\sigma_1,\eta,\sigma_2} P_{\mu^\vee/\sigma_1{}^\vee}\left(-y^1;t,q\right) P_{\eta/\sigma_1}\left(\iota x^1;q,t\right) P_{\eta^\vee/\sigma_2{}^\vee}\left(-\iota y^2;t,q\right) P_{\nu/\sigma_2}\left(x^2;q,t\right) v^{-|\sigma_1|+|\sigma_2|}Q^{|\eta|}.
\end{aligned}\tag{5.11}$$

From (5.8), they reduce to

$$\begin{aligned}
Z_{\mu\lambda_1\lambda_2}{}^{\nu}(Q;q,t) =& P_{\lambda_1}(t^\rho;q,t)\,P_{\lambda_2}(t^\rho;q,t)\,f_\nu(q,t)^{-1}\,v^{-|\nu|}\\
&\times\sum_\eta \iota P_{\mu^\vee/\eta^\vee}\left(-y^1,-Qy^2;t,q\right) P_{\nu/\eta}\left(Qx^1,x^2;q,t\right)(vQ)^{|\eta|}\Pi_0\left(-v^{-1}Qx^1,y^2\right)^{-1},\\
Z_{\mu\lambda_1}{}^{\lambda_2\nu}(Q;q,t) =& P_{\lambda_1}(t^\rho;q,t)\,P_{\lambda_2{}^\vee}(-q^\rho;t,q)\\
&\times\sum_\eta P_{\mu^\vee/\eta^\vee}\left(-\iota y^1,-vQy^2;t,q\right) P_{\nu/\eta}\left(v^{-1}Qx^1,\iota x^2;q,t\right) Q^{|\eta|}\Pi_0\left(-Qx^1,y^2\right),\\
Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t) =& P_{\lambda_1{}^\vee}(-q^\rho;t,q)\,P_{\lambda_2}(t^\rho;q,t)\,f_\mu(q,t)\,f_\nu(q,t)^{-1}\,v^{|\mu|-|\nu|}\\
&\times\sum_\eta P_{\mu^\vee/\eta^\vee}\left(-y^1,-v^{-1}Q\iota y^2;t,q\right) P_{\nu/\eta}\left(vQ\iota x^1,x^2;q,t\right) Q^{|\eta|}\Pi_0\left(-Qx^1,y^2\right).
\end{aligned}\tag{5.12}$$

Since $\Pi_0\left(-Qx^1,y^2\right)/\Pi_0\left(-Qt^\rho,q^\rho\right) = N_{\lambda_1\lambda_2}(vQ;q,t)$, the instanton part, such as $Z_{\mu\lambda_1\lambda_2}{}^{\nu}(Q;q,t)/Z_{\bullet\bullet\bullet}{}^{\bullet}(Q;q,t)$ is written not by $\Pi_0\left(-v^{-1}Qx^1,y^2\right)$'s but by $N_{\lambda_1\lambda_2}(Q;q,t)$'s.

Note that

$$Z_{\mu\lambda_1}{}^{\lambda_2\nu}(Q;q,t) = Z_{\nu\lambda_2}{}^{\lambda_1\mu}(Q;q^{-1},t^{-1}) = Z_{\nu^\vee\lambda_2{}^\vee}{}^{\lambda_1{}^\vee\mu^\vee}(Q;t,q)(-1)^{|\lambda_1|+|\lambda_2|+|\mu|+|\nu|}. \tag{5.13}$$

5.4 Flop operation

The flop invariance of the topological vertex is shown in [32, 34]. We can show the flop invariance of the refined topological vertex as follows. First,

$$Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t) = Z_{\mu\lambda_2}{}^{\lambda_1\nu}(Q^{-1};q,t)Q^{|\mu|+|\nu|}\frac{\Pi_0\left(-Qx^1,y^2\right)}{\Pi_0\left(-Q^{-1}x^2,y^1\right)}\frac{f_\mu\left(q,t\right)}{f_\nu\left(q,t\right)}. \tag{5.14}$$

Next, from (2.33) and (2.34), we have

$$\begin{aligned}\frac{\Pi_0\left(-Qx^1,y^2\right)/\Pi_0\left(-Qt^\rho,q^\rho\right)}{\Pi_0\left(-Q^{-1}x^2,y^1\right)/\Pi_0\left(-Q^{-1}t^\rho,q^\rho\right)} &= \frac{N_{\lambda_1\lambda_2}\left(vQ;q,t\right)}{N_{\lambda_2\lambda_1}\left(vQ^{-1};q,t\right)} \\ &= \frac{N_{\lambda_1\lambda_2}\left(vQ;q,t\right)}{N_{\lambda_1{}^\vee\lambda_2{}^\vee}\left(v^{-1}Q^{-1};t,q\right)} = Q^{|\lambda_1|+|\lambda_2|}\frac{f_{\lambda_1}\left(q,t\right)}{f_{\lambda_2}\left(q,t\right)}.\end{aligned} \tag{5.15}$$

Thus we obtain the following flop invariance[9]

$$\begin{aligned}\frac{Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}(Q;q,t)}{Z_{\mu'}{}^{\bullet}{}_{\bullet}{}^{\nu'}(Q;q,t)} = \frac{Z_{\mu\lambda_2}{}^{\lambda_1\nu}(Q^{-1};q,t)}{Z_{\mu'\bullet}{}^{\bullet\nu'}(Q^{-1};q,t)}Q^{|\lambda_1|+|\lambda_2|+|\mu|+|\nu|}&\frac{f_\mu\left(q,t\right)}{f_\nu\left(q,t\right)}\frac{f_{\lambda_1}\left(q,t\right)}{f_{\lambda_2}\left(q,t\right)} \\ \times Q^{-|\mu'|-|\nu'|}&\frac{f_{\nu'}\left(q,t\right)}{f_{\mu'}\left(q,t\right)}.\end{aligned} \tag{5.16}$$

The denominator corresponds to the perturbative part.

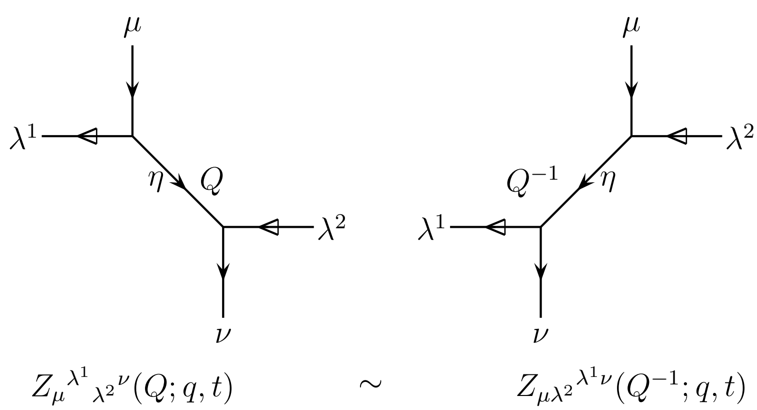


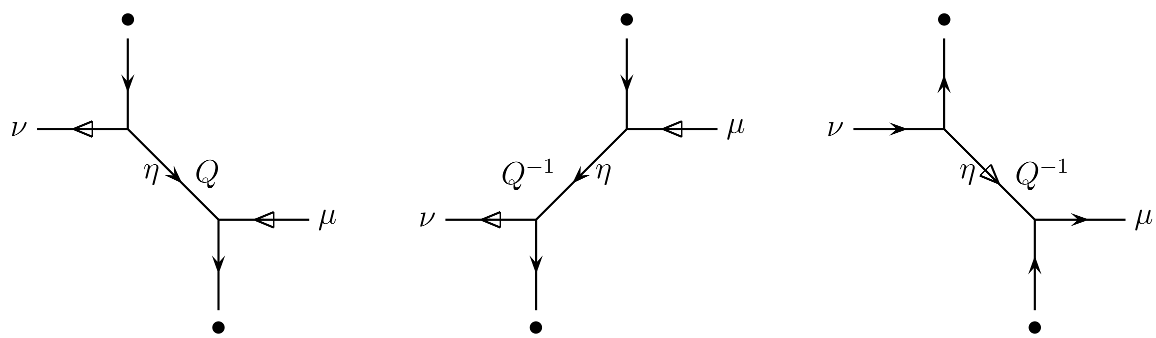
Figure 7: Flop invariance

Combining (5.16) with (5.4), we have

$$\frac{Z_{\bullet}{}^{\nu}{}_{\mu}{}^{\bullet}(Q;q,t)}{Z_{\bullet}{}^{\bullet}{}_{\bullet}{}^{\bullet}(Q;q,t)}Q^{-|\mu|-|\nu|} = \frac{Z_{\bullet\mu}{}^{\nu\bullet}(Q^{-1};q,t)}{Z_{\bullet\bullet}{}^{\bullet\bullet}(Q^{-1};q,t)}\frac{f_\nu\left(q,t\right)}{f_\mu\left(q,t\right)} = \frac{Z^{\bullet}{}_{\nu\bullet}{}^{\mu}(Q^{-1};q,t)}{Z^{\bullet}{}_{\bullet\bullet}{}^{\bullet}(Q^{-1};q,t)}v^{|\mu|-|\nu|}, \tag{5.17}$$

which changes the preferred direction also.

[9] The flop invariance of $C^{(IKV)}_{\mu\nu\lambda}(t,q)$ has recently been discussed in [39].



$$Z_{\bullet}{}^{\nu}{}_{\mu}{}^{\bullet}(Q;q,t) \qquad \sim \qquad Z_{\bullet\mu}{}^{\nu\bullet}(Q^{-1};q,t) \quad \sim \quad Z^{\bullet}{}_{\nu\bullet}{}^{\mu}(Q^{-1};q,t)$$

Figure 8: Changing the preferred direction II

5.5 Finite N Macdonald polynomial and homological invariants

When λ_1 or λ_2 is the trivial representation, the amplitudes of the above diagrams are written by the Macdonald polynomials with a finite number of variables. Note that $P_\lambda\left(ax^1, b\iota x^2; q, t\right)$ with $x^\alpha = q^{\lambda_\alpha} t^\rho$ and $a, b \in \mathbb{C}$ is the Macdonald function in the power sum functions

$$p_n\left(ax^1\right) + \iota p_n\left(bx^2\right) = \sum_{i=1}^{\infty} \left\{a^n\left(q^{n\lambda_{1,i}} - 1\right) - b^n\left(q^{n\lambda_{2,i}} - 1\right)\right\} t^{n(\frac{1}{2}-i)} + \frac{a^n - b^n}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}. \quad (5.18)$$

For $N \in \mathbb{N}$ and $N \geq \ell(\lambda)$,

$$p_n\left(q^\lambda t^\rho\right) + \iota p_n\left(t^{-N+\rho}\right) = \sum_{i=1}^{\infty}\left(q^{n\lambda_i} - 1\right) t^{n(\frac{1}{2}-i)} + \frac{1 - t^{-nN}}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}} = \sum_{i=1}^{N}\left(q^{\lambda_i} t^{\frac{1}{2}-i}\right)^n, \quad (5.19)$$

which are the power sum symmetric polynomials in N variables. Therefore $P_\lambda\left(q^\lambda t^\rho, t^{-N-\rho}; q, t\right)$ is the Macdonald polynomial in N variables $\{q^{\lambda_i} t^{\frac{1}{2}-i}\}_{1\leq i\leq N}$. On the other hand, from (E.7) and (B.21)

$$\begin{aligned}
N_{\lambda\bullet}(vQ;q,t) &= \prod_{(i,j)\in\lambda}\left(1 - vQq^{\lambda_i - j}t^{1-i}\right) = \prod_{(i,j)\in\lambda}\left(1 - vQq^{j-1}t^{1-i}\right) \\
&= \frac{P_{\lambda^\vee}(q^\rho, vQq^{-\rho}; t, q)}{P_{\lambda^\vee}(q^\rho; t, q)} = \frac{P_\lambda(t^\rho, v^{-1}Q^{-1}t^{-\rho}; q, t)}{P_\lambda(t^\rho; q, t)} Q^{|\lambda|} f_\lambda(q,t), \\
N_{\bullet\lambda}(vQ;q,t) &= \prod_{(i,j)\in\lambda}\left(1 - vQq^{-\lambda_i + j - 1}t^{i}\right) = \prod_{(i,j)\in\lambda}\left(1 - v^{-1}Qq^{1-j}t^{i-1}\right) \\
&= \frac{P_\lambda(t^\rho, v^{-1}Qt^{-\rho}; q, t)}{P_\lambda(t^\rho; q, t)} = \frac{P_{\lambda^\vee}(q^\rho, vQ^{-1}q^{-\rho}; t, q)}{P_{\lambda^\vee}(q^\rho; t, q)} Q^{|\lambda|} f_\lambda(q,t)^{-1}. \quad (5.20)
\end{aligned}$$

Therefore, some factors in $Z_{\mu\lambda_1}{}^{\lambda_2\nu}/Z_{\bullet\bullet}{}^{\bullet\bullet}$ and $Z_{\mu}{}^{\lambda_1}{}_{\lambda_2}{}^{\nu}/Z_{\bullet}{}^{\bullet}{}_{\bullet}{}^{\bullet}$ might be written by the Macdonald polynomials in a finite number of variables. For example, if μ and one of the λ_α ($\alpha = 1$ or 2) are the trivial representation,

$$\frac{Z_{\bullet\lambda}{}^{\bullet\nu}(Q^{-1};q,t)}{Z_{\bullet\bullet}{}^{\bullet\bullet}(Q^{-1};q,t)} = P_\nu\left(q^\lambda t^\rho, vQt^{-\rho};q,t\right) P_\lambda\left(t^\rho, v^{-1}Qt^{-\rho};q,t\right) v^{-|\nu|}Q^{-|\lambda|-|\nu|} f_\lambda(q,t),$$
$$\frac{Z_{\bullet}{}^{\bullet}{}_{\lambda}{}^{\nu}(Q;q,t)}{Z_{\bullet\bullet}{}^{\bullet\bullet}(Q;q,t)} = P_\nu\left(q^\lambda t^\rho, vQt^{-\rho};q,t\right) P_\lambda\left(t^\rho, v^{-1}Qt^{-\rho};q,t\right) v^{-|\nu|} f_\nu(q,t)^{-1}. \tag{5.21}$$

When $v^{\pm1}Q = t^{-N}$ with $N \in \mathbb{N}$, they are written by the Macdonald polynomials in N variables. These are candidates for the $SU(N)$ homological invariants.

Note that

$$\mathcal{W}_{\lambda,\nu}(q,t) := C_{\bullet\lambda}{}^{\nu}(q,t)v^{|\nu|}f_\nu(q,t) = P_\lambda(t^\rho;q,t)\, P_\nu\left(q^\lambda t^\rho;q,t\right), \tag{5.22}$$

has a nice symmetry [15](Ch. VI.6):

$$\mathcal{W}_{\lambda,\nu}(q,t) = \mathcal{W}_{\nu,\lambda}(q,t). \tag{5.23}$$

When $t = q$, $\mathcal{W}_{\lambda,\nu}(q,q)$ gives a large N limit of the Hopf link invariants.

6 One-Loop Diagrams

Some one-loop diagrams which correspond to the trace of the vertex operators can be calculated by the following trace formula.

6.1 Trace formula

First, we have:

Lemma. Let x and y be sets of variables and a, b and $c := ab \in \mathbb{C}$. If $|c| < 1$, then

$$\sum_{\lambda,\mu} P_{\lambda^\vee/\mu^\vee}(x;t,q)\, P_{\lambda/\mu}(y;q,t)\, a^{|\lambda|}b^{|\mu|} = \prod_{k\geq 0} \frac{\Pi_0\left(ac^k x, y\right)}{1-c^{k+1}}$$
$$= \exp\left\{-\sum_{n>0}\frac{1}{n}\frac{p_n(ax)p_n(-y)-c^n}{1-c^n}\right\}. \tag{6.1}$$

Proof. As in [15](Ch. I.5), let $F(x,y)$ denote the left-hand side of the above equation. Then it follows from the Cauchy formula (5.6) that

$$F(x,y) = \sum_{\lambda,\mu} P_{\lambda^\vee/\mu^\vee}(ax;t,q)\, P_{\lambda/\mu}(y;q,t)\,(ab)^{|\mu|}$$

$$
\begin{aligned}
&= \sum_{\lambda,\mu} P_{\mu^\vee/\lambda^\vee}(ax;t,q)\, P_{\mu/\lambda}(y;q,t)\,(ab)^{|\mu|}\Pi_0(ax,y) \\
&= \sum_{\lambda,\mu} P_{\mu^\vee/\lambda^\vee}(abx;t,q)\, P_{\mu/\lambda}(y;q,t)\,a^{|\mu|}b^{|\lambda|}\Pi_0(ax,y)\,.
\end{aligned} \tag{6.2}
$$

Therefore

$$
F(x,y) = F(cx,y)\Pi_0(ax,y) = F(0,y)\prod_{k\geq 0}\Pi_0\left(ac^k x, y\right), \qquad |c|<1. \tag{6.3}
$$

But

$$
F(0,y) = \sum_{\lambda} P_{\lambda/\lambda}(y;q,t)c^{|\lambda|} = \sum_{\lambda} c^{|\lambda|} = \prod_{n>0}(1-c^n)^{-1}, \qquad |c|<1, \tag{6.4}
$$

and the lemma is proven. □

From the above lemma and (5.10) we obtain the following trace formula.

Proposition. For $N\in\mathbb{N}$, let $x^i = x^{2N+i}$'s be sets of variables, $\lambda_0=\lambda_{2N}$, $c_{i,i+1} = c_{2N+i,2N+i+1}\in\mathbb{C}$, $c_{i,j} := \prod_{k=i}^{j-1} c_{i,i+1}$ and $c := c_{1,2N+1} = \prod_{i=1}^{2N} c_{i,i+1}$. If $|c|<1$, then

$$
\begin{aligned}
&\sum_{\{\lambda_1,\lambda_2,\cdots,\lambda_{2N}\}} \prod_{i=1}^{N} P_{\lambda_{2i-2}{}^\vee/\lambda_{2i-1}{}^\vee}\left(x^{2i-1};t,q\right) P_{\lambda_{2i}/\lambda_{2i-1}}\left(x^{2i};q,t\right)\cdot \prod_{i=1}^{2N} c_{i,i+1}^{|\lambda_i|} \\
&= \prod_{k\geq 0}\frac{1}{1-c^{k+1}} \prod_{i=1}^{N}\prod_{j=i+1}^{i+N} \Pi_0\left(x^{2i},\ c_{2i,2j-1}c^k x^{2j-1}\right) \\
&= \exp\left\{-\sum_{n>0}\frac{1}{n}\frac{1}{1-c^n}\left\{\sum_{i=1}^{N}\sum_{j=i+1}^{i+N} c_{2i,2j-1}^n p_n\left(x^{2i}\right)p_n\left(-x^{2j-1}\right) - c^n\right\}\right\}.
\end{aligned} \tag{6.5}
$$

Proof. From (5.10) and (6.1), the left-hand side of the above equation is

$$
\begin{aligned}
&\sum_{\lambda,\eta} P_{\lambda^\vee/\eta^\vee}\left(\{c_{1,2i-1}x^{2i-1}\}_{i=1}^N;t,q\right) P_{\lambda/\eta}\left(\{x^{2i}c_{2i,2N}\}_{i=1}^N;q,t\right) c_{1,2N}^{|\eta|}c_{2N,2N+1}^{|\lambda|} \\
&\qquad\qquad \times \prod_{1\leq i<j\leq N}\Pi_0\left(x^{2i},\ c_{2i,2j-1}x^{2j-1}\right) \\
&= \prod_{1\leq i<j\leq N}\Pi_0\left(x^{2i},\ c_{2i,2j-1}x^{2j-1}\right)\cdot \prod_{k\geq 0}\frac{\Pi_0\left(\{x^{2i}c_{2i,2N+1}\}_{i=1}^N, \{c_{1,2i-1}c^k x^{2i-1}\}_{i=1}^N\right)}{1-c^{k+1}},
\end{aligned} \tag{6.6}
$$

here $\Pi_0\left(\{x^i\}_{i=1}^N, \{y^j\}_{j=1}^M\right) = \prod_{i=1}^N\prod_{j=1}^M \Pi_0(x^i,y^j)$. Then the left-hand side of (6.5) reduces to

$$
\prod_{1\leq i<j\leq N}\Pi_0\left(x^{2i},\ c_{2i,2j-1}x^{2j-1}\right)\cdot\prod_{k\geq 0}\frac{1}{1-c^{k+1}}\prod_{i,j=1}^{N}\Pi_0\left(x^{2i},\ c_{2i,2N+2j-1}c^k x^{2N+2j-1}\right)
$$

$$=\prod_{k\geq 0}\frac{1}{1-c^{k+1}}\prod_{1\leq i<j\leq N}\Pi_0\left(x^{2i},\ c_{2i,2j-1}c^k x^{2j-1}\right)\prod_{1\leq j\leq i\leq N}\Pi_0\left(x^{2i},\ c_{2i,2N+2j-1}c^k x^{2j-1}\right),\tag{6.7}$$

which equals to the second line of (6.5). □

From this trace formula, we can calculate one-loop diagram if the loop does not contain preferred directions and also the framing factors cancel out.

6.2 Examples for $N=2$ and 4

For an example of the trace formula for $N=2$, let

$$\begin{aligned}Z_2 :=& \sum_{\mu,\nu} C^{\nu}{}_{\bullet\lambda}(q,t)C_{\nu}{}^{\bullet\lambda}(q,t)\Lambda^{|\lambda|}Q^{|\nu|}\\ =& \sum_{\mu,\nu,\sigma_1,\sigma_2} P_{\nu^\vee/\sigma_1{}^\vee}\left(-\iota q^\rho;t,q\right)P_{\lambda/\sigma_1}\left(t^\rho;t,q\right)P_{\lambda^\vee/\sigma_2{}^\vee}\left(-q^\rho;t,q\right)P_{\nu/\sigma_2}\left(\iota t^\rho;t,q\right)v^{|\sigma_1|-|\sigma_2|}\Lambda^{|\lambda|}Q^{|\nu|}.\end{aligned}\tag{6.8}$$

Then from (6.5) with $(c_{1,2},c_{2,3},c_{3,4},c_{4,5})=(v,\Lambda,v^{-1},Q)$ and $(x^1,x^2,x^3,x^4)=(-\iota q^\rho,t^\rho,-q^\rho,\iota t^\rho)$, it follows that $c=Q\Lambda$ and

$$\begin{pmatrix} c_{2,3} & c_{2,5}\\ c_{4,5} & c_{4,7}\end{pmatrix}=\begin{pmatrix}\Lambda & c/v\\ Q & cv\end{pmatrix},\tag{6.9}$$

and thus

$$Z_2=\prod_{k\geq 0}\frac{\Pi_0\left(t^\rho,-\Lambda c^k q^\rho\right)\Pi_0\left(t^\rho,-Qc^k q^\rho\right)}{\Pi_0\left(t^\rho,-vc^{k+1}q^\rho\right)\Pi_0\left(t^\rho,-v^{-1}c^{k+1}q^\rho\right)}\frac{1}{1-c^{k+1}}.\tag{6.10}$$

From (1.8), we obtain

$$Z_2=\exp\left\{-\sum_{n>0}\frac{1}{n}\frac{1}{1-c^n}\left\{\frac{(\Lambda^n+Q^n)-(v^n+v^{-n})c^n}{(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(q^{\frac{n}{2}}-q^{-\frac{n}{2}})}-c^n\right\}\right\}.\tag{6.11}$$

If we separate out the part $Z_2^{\text{pert}}:=Z_2(\Lambda=0)=\exp\left\{-\sum_{n>0}Q^n/(n(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(q^{\frac{n}{2}}-q^{-\frac{n}{2}}))\right\}$, then $Z_2^{\text{inst}}:=Z_2/Z_2^{\text{pert}}$ is

$$Z_2^{\text{inst}}=\exp\left\{-\sum_{n>0}\frac{1}{n}\frac{\Lambda^n}{1-c^n}\frac{(Q^n-u^n)(Q^n-u^{-n})}{(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(q^{\frac{n}{2}}-q^{-\frac{n}{2}})}\right\}.\tag{6.12}$$

As we will see in section 7.2, this gives the equivariant χ_y genus of the Hilbert scheme of points on $\mathbb{C}^2$.

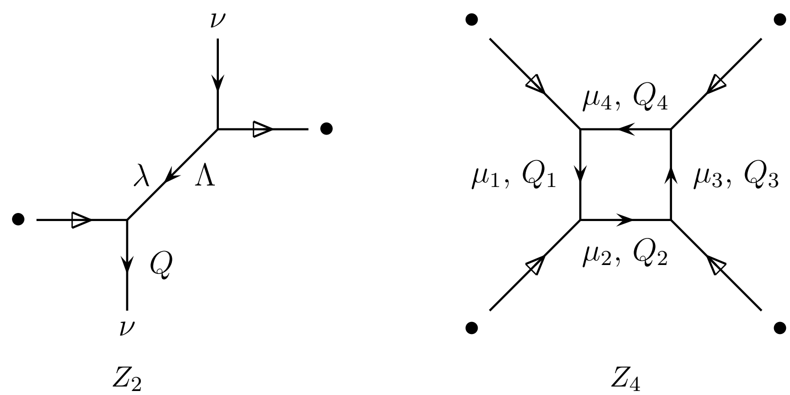


Figure 9: Examples for the trace formula: the framing indices for the internal lines of Z_4 are all 1.

For an example for $N=4$, let

$$\begin{aligned} Z_4 :=& \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{4} C_{\mu_\alpha \bullet}{}^{\mu_{\alpha+1}}(q,t) f_{\mu_{\alpha+1}}(q,t) \\ =& \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{4} P_{\mu_\alpha{}^\vee/\sigma_\alpha{}^\vee}(-\iota q^\rho) P_{\mu_{\alpha+1}{}^\vee/\sigma_\alpha{}^\vee}(t^\rho) v^{|\sigma_\alpha|-|\mu_\alpha|} Q^{|\mu_\alpha|}, \end{aligned} \tag{6.13}$$

with $\mu_5=\mu_1$. Then from (6.5) with $(c_{2\alpha-1,2\alpha}, c_{2\alpha,2\alpha+1}) = (v, v^{-1}Q_\alpha)$ and $(x^{2\alpha}, x^{2\alpha-1}) = (t^\rho, -\iota q^\rho)$, it follows that

$$Z_4 = \prod_{k\geq 0} \frac{1}{1-c^{k+1}} \prod_{i=1}^{4} \prod_{j=i+1}^{i+4} \Pi_0\left(t^\rho,\ -c^k c_{2i,2j-1} q^\rho\right)^{-1}, \tag{6.14}$$

where $c=Q_1Q_2Q_3Q_4$ and

$$\begin{pmatrix} c_{2,3} & c_{2,5} & c_{2,7} & c_{2,9} \\ c_{4,5} & c_{4,7} & c_{4,9} & c_{4,11} \\ c_{6,7} & c_{6,9} & c_{6,11} & c_{6,13} \\ c_{8,9} & c_{8,11} & c_{8,13} & c_{8,15} \end{pmatrix} = v^{-1} \begin{pmatrix} Q_1 & Q_1Q_2 & Q_1Q_2Q_3 & c \\ Q_2 & Q_2Q_3 & Q_2Q_3Q_4 & c \\ Q_3 & Q_3Q_4 & Q_3Q_4Q_1 & c \\ Q_4 & Q_4Q_1 & Q_4Q_1Q_2 & c \end{pmatrix}. \tag{6.15}$$

Thus

$$Z_4 = \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{1}{1-c^n} \left\{ \frac{v^{-n}\sum_{\alpha=1}^{4}\left(Q_\alpha^n + Q_\alpha^n Q_{\alpha+1}^n + Q_\alpha^n Q_{\alpha+1}^n Q_{\alpha+2}^n + c^n\right)}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(q^{\frac{n}{2}} - q^{-\frac{n}{2}})} - c^n \right\} \right\}, \tag{6.16}$$

where $Q_{i+4}=Q_i$.

7 $U(1)$ Partition Function, χ_y Genus and Elliptic Genus

Nekrasov's $U(1)$ partition function, the χ_y genus and the elliptic genus are realized by our refined topological vertex, as shown in [14]. Since the diagrams for $U(1)$ theory have trivial framing, the vertex in [14] and the improved vertex in the present paper give the same answer.

7.1 $U(1)$ partition function

First, the $U(1)$ partition function is written as follows. Let

$$Z := \sum_{\lambda} \Lambda^{|\lambda|} C_{\bullet\lambda}{}^{\bullet}(q,t) C^{\bullet\lambda}{}_{\bullet}(q,t). \tag{7.1}$$

Then

$$\begin{aligned} Z &= \sum_{\lambda} \Lambda^{|\lambda|} P_{\lambda}(t^{\rho};q,t)\ P_{\lambda^\vee}(-q^{\rho};t,q) \\ &= \sum_{\lambda} \prod_{s\in\lambda} v^{-1}\Lambda \frac{1}{(1-q^{a(s)}t^{\ell(s)+1})(1-q^{-a(s)-1}t^{-\ell(s)})}, \end{aligned} \tag{7.2}$$

from (4.3). This agrees with the $U(1)$ Nekrasov's formula $Z_0^{\text{inst}}\left(\mathbf{e}_1, \Lambda^{\frac{1}{2}}; q, t\right)$ in (2.4). Using the Cauchy-formula (5.5) we have

$$\begin{aligned} Z &= \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(q^{\frac{n}{2}}-q^{-\frac{n}{2}})} \right\} \\ &= \exp\left\{ \mp \sum_{n>0} \frac{1}{n} \sum_{i,j} \left(\Lambda\, t^{\frac{1}{2}-i} q^{\pm(\frac{1}{2}-j)} \right)^n \right\}, \qquad |q^{\mp1}|, |t^{-1}| < 1 \\ &= \prod_{i,j\geq 1} (1-\Lambda\, t^{\frac{1}{2}-i} q^{\pm(\frac{1}{2}-j)})^{\pm 1}, \qquad |q^{\mp1}|, |t^{-1}| < 1. \end{aligned} \tag{7.3}$$

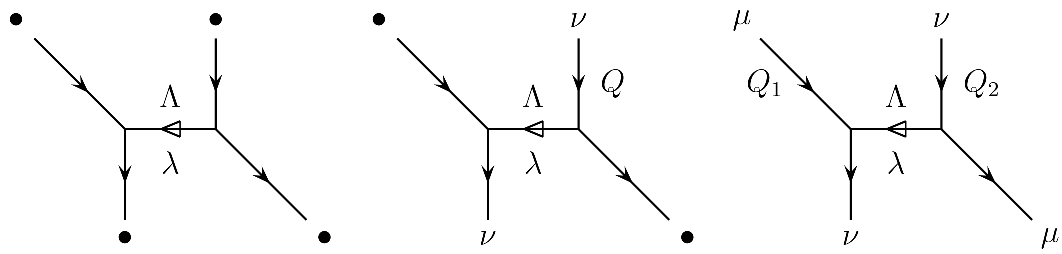
7.2 χ_y genus

Next, the χ_y genus is realized as follows. Let

$$\widetilde{Z} := \sum_{\lambda,\nu} Q^{|\nu|} \Lambda^{|\lambda|} C_{\bullet\lambda}{}^{\nu}(q,t) C^{\bullet\lambda}{}_{\nu}(q,t). \tag{7.4}$$

Then

$$\widetilde{Z} = \sum_{\lambda,\nu} Q^{|\nu|} \Lambda^{|\lambda|} P_{\lambda}\left(t^{\rho};q,t\right) P_{\lambda^\vee}\left(-q^{\rho};t,q\right) P_{\nu}\left(q^{\lambda}t^{\rho};q,t\right) P_{\nu^\vee}\left(-t^{\lambda^\vee}q^{\rho};t,q\right). \tag{7.5}$$



$U(1)$ partition function $\qquad$ χ_y genus $\qquad$ Elliptic genus

Figure 10: $U(1)$ partition function, χ_y genus and Elliptic genus

From (4.3) and (5.5) we have

$$\widetilde{Z} = \sum_{\lambda} \Pi_0(-Qq^{\lambda}t^{\rho},\, t^{\lambda^\vee}q^{\rho}) \prod_{s\in\lambda} v^{-1}\Lambda \frac{1}{(1-q^{a(s)}t^{\ell(s)+1})(1-q^{-a(s)-1}t^{-\ell(s)})}. \tag{7.6}$$

If we separate out the part $\widetilde{Z}^{\mathrm{pert}} := \sum_\nu Q^{|\nu|} C_{\bullet\bullet}{}^{\nu}(q,t) C^{\bullet\bullet}{}_{\nu}(q,t) = \Pi_0(-Qt^{\rho},\, q^{\rho})$, which is independent of Λ, then $Z^{\mathrm{inst}} := \widetilde{Z}/\widetilde{Z}^{\mathrm{pert}}$ is from (2.10)

$$\begin{aligned} Z^{\mathrm{inst}} &= \sum_{\lambda} \left(v^{-1}\Lambda\right)^{|\lambda|} \frac{N_{\lambda\lambda}(vQ;q,t)}{N_{\lambda\lambda}(1;q,t)} \\ &= \sum_{\lambda}\prod_{s\in\lambda} v^{-1}\Lambda \frac{1-vQq^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)}t^{\ell(s)+1}} \frac{1-vQq^{-a(s)-1}t^{-\ell(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}. \end{aligned} \tag{7.7}$$

This agrees with the χ_y genus (20) of [35] with $vQ = y$, $v^{-1}\Lambda = Q^{\mathrm{LLZ}}$ and $(q,t) = (1/t_1, t_2)$ or $(1/t_2, t_1)$.

If our refined topological vertex had cyclic symmetry, then this χ_y genus Z^{inst} would agree with Z_2^{inst} in section 6.2, and hence the following identity should hold

$$\begin{aligned} &\sum_{\lambda} \Lambda^{|\lambda|} \prod_{s\in\lambda} \frac{1-Qq^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)}t^{\ell(s)+1}} \frac{1-Qq^{-a(s)-1}t^{-\ell(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}} \\ &\qquad = \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1-\Lambda^n Q^n} \frac{(1-t^nQ^n)(1-q^{-n}Q^n)}{(1-t^n)(1-q^{-n})} \right\}. \end{aligned} \tag{7.8}$$

From (B.21), this is close to the Cauchy formula for the Macdonald functions in power sums $p_n = (1-t^nQ^n)/(1-t^n)$ and $(-\Lambda)^n(1-q^{-n}Q^n)/(1-q^{-n})$, i.e.

$$\sum_{\lambda} (-\Lambda)^{|\lambda|} \prod_{s\in\lambda} \frac{1-Qq^{a'(s)}t^{1-\ell'(s)}}{1-q^{a(s)}t^{\ell(s)+1}} \frac{1-Qq^{a'(s)-1}t^{-\ell'(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}} q^{-a'(s)}t^{\ell'(s)}$$

$$= \exp\left\{ - \sum_{n>0} \frac{1}{n} \Lambda^n \frac{(1-t^n Q^n)(1-q^{-n}Q^n)}{(1-t^n)(1-q^{-n})} \right\}. \tag{7.9}$$

Although we have no proof for (7.8), computer calculations support that $Z^{\mathrm{inst}} = Z_2^{\mathrm{inst}}$, which strongly suggests a kind of symmetry of web diagrams. See also the discussions in the recent papers [18, 36].

7.3 Elliptic genus

Finally, the elliptic genus is written as follows. Let

$$\widetilde{Z} := \sum_{\lambda,\mu,\nu} Q_1^{|\mu|} \Lambda^{|\lambda|} Q_2^{|\nu|} C_{\mu\lambda}{}^{\nu}(q,t) C^{\mu\lambda}{}_{\nu}(q,t). \tag{7.10}$$

Then

$$\begin{aligned} \widetilde{Z} = &\sum_{\lambda,\mu,\nu} P_\lambda(t^\rho;q,t) \sum_\sigma \iota P_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee} q^\rho;t,q)\ P_{\nu/\sigma}(q^\lambda t^\rho;q,t) Q_1^{|\mu|} \Lambda^{|\lambda|} Q_2^{|\nu|} \\ &\times P_{\lambda^\vee}(-q^\rho;t,q) \sum_\eta \iota P_{\mu/\eta}(q^\lambda t^\rho;q,t)\ P_{\nu^\vee/\eta^\vee}(-t^{\lambda^\vee} q^\rho;t,q) v^{|\sigma|-|\eta|}. \end{aligned} \tag{7.11}$$

From (4.3) and the trace formula (6.5) with $(c_{1,2}, c_{2,3}, c_{3,4}, c_{4,5}) = (v, Q_2, v^{-1}, Q_1)$ and $(x^1, x^2, x^3, x^4) = (-\iota t^{\lambda^\vee} q^\rho, q^\lambda t^\rho, -t^{\lambda^\vee} q^\rho, \iota q^\lambda t^\rho)$, it follows that

$$\begin{aligned} \widetilde{Z} = &\sum_\lambda \prod_{s\in\lambda} v^{-1}\Lambda \frac{1}{(1-q^{a(s)}t^{\ell(s)+1})(1-q^{-a(s)-1}t^{-\ell(s)})} \\ &\times \prod_{k\geq 0} \frac{\Pi_0(-Q_1 c^k q^\lambda t^\rho,\ t^{\lambda^\vee} q^\rho)\ \Pi_0(-Q_2 c^k q^\lambda t^\rho,\ t^{\lambda^\vee} q^\rho)}{\Pi_0(-v^{-1} c^{k+1} q^\lambda t^\rho,\ t^{\lambda^\vee} q^\rho) \Pi_0(-v c^{k+1} q^\lambda t^\rho,\ t^{\lambda^\vee} q^\rho)} \frac{1}{1-c^{k+1}}, \end{aligned} \tag{7.12}$$

with $c = Q_1 Q_2$ and $|c| < 1$. If we factor out the Λ-independent part

$$\begin{aligned} \widetilde{Z}^{\mathrm{pert}} :=& \sum_{\mu,\nu} Q_1^{|\mu|} Q_2^{|\nu|} C_{\mu\bullet}{}^{\nu}(q,t) C^{\mu\bullet}{}_{\nu}(q,t) \\ =& \prod_{k\geq 0} \frac{\Pi_0(-Q_1 c^k t^\rho,\ q^\rho)\ \Pi_0(-Q_2 c^k t^\rho,\ q^\rho)}{\Pi_0(-v^{-1} c^{k+1} t^\rho,\ q^\rho) \Pi_0(-v c^{k+1} t^\rho,\ q^\rho)} \frac{1}{1-c^{k+1}}, \end{aligned} \tag{7.13}$$

then $Z^{\mathrm{inst}} := \widetilde{Z}/\widetilde{Z}^{\mathrm{pert}}$ is from (2.10),

$$\begin{aligned} Z^{\mathrm{inst}} =& \sum_\lambda \left(v^{-1}\Lambda\right)^{|\lambda|} \prod_{k\geq 0} \frac{N_{\lambda\lambda}\left(vQ_1 c^k;q,t\right) N_{\lambda\lambda}\left(vQ_2 c^k;q,t\right)}{N_{\lambda\lambda}\left(c^k;q,t\right) N_{\lambda\lambda}\left(v^2 c^{k+1};q,t\right)} \\ =& \sum_\lambda \prod_{k\geq 0} \prod_{s\in\lambda} v^{-1}\Lambda \frac{\left(1-vQ_1^{k+1}Q_2^k q^{a(s)} t^{\ell(s)+1}\right)\left(1-vQ_1^k Q_2^{k+1} q^{a(s)} t^{\ell(s)+1}\right)}{\left(1-Q_1^k Q_2^k q^{a(s)} t^{\ell(s)+1}\right)\left(1-v^2 Q_1^{k+1} Q_2^{k+1} q^{a(s)} t^{\ell(s)+1}\right)} \end{aligned}$$

$$\times \frac{\left(1 - vQ_1^{k+1}Q_2^{k}q^{-a(s)-1}t^{-\ell(s)}\right)\left(1 - vQ_1^{k}Q_2^{k+1}q^{-a(s)-1}t^{-\ell(s)}\right)}{\left(1 - Q_1^{k}Q_2^{k}q^{-a(s)-1}t^{-\ell(s)}\right)\left(1 - v^2Q_1^{k+1}Q_2^{k+1}q^{-a(s)-1}t^{-\ell(s)}\right)}. \quad (7.14)$$

This agrees with the elliptic genus (24) of [35] with $Q_1Q_2 = p$, $vQ_1 = y$, $v^{-1}\Lambda = y^{-1}Q^{\mathrm{LLZ}}$ and $(q,t) = (t_1, 1/t_2)$ or $(t_2, 1/t_1)$.

8 $SU(N_c)$ Partition Function

Nekrasov's $SU(N_c)$ partition function is also realized by our refined topological vertex, as mentioned in [14].

8.1 Pure $SU(2)$ partition function

The pure $SU(2)$ partition function without Chern-Simons couplings is written as follows. Let

$$\begin{aligned} Z^{\lambda_1,\lambda_2}_{\mathbf{e}_1,\mathbf{e}_2}(q,t) &:= \sum_{\mu} C_{\bullet\lambda_1}{}^{\mu}(q,t) C_{\mu\lambda_2}{}^{\bullet}(q,t) Q_{1,2}^{|\mu|} f_\mu(q,t) \\ &= \sum_{\mu} P_{\lambda_1}(t^\rho;q,t)\, P_\mu\left(q^{\lambda_1}t^\rho;q,t\right) \iota P_{\mu^\vee}\left(-t^{\lambda_2^\vee}q^\rho;t,q\right) P_{\lambda_2}(t^\rho;q,t)\left(v^{-1}Q_{1,2}\right)^{|\mu|} \\ &= \Pi_0\left(-v^{-1}Q_{1,2}\, q^{\lambda_1}t^\rho,\ t^{\lambda_2^\vee}q^\rho\right)^{-1} P_{\lambda_1}(t^\rho;q,t)\, P_{\lambda_2}(t^\rho;q,t), \end{aligned} \quad (8.1)$$

from (5.5), where $Q_{\alpha,\beta} := \mathbf{e}_\alpha/\mathbf{e}_\beta$. The dual part is

$$\begin{aligned} Z^{\lambda_2^\vee,\lambda_1^\vee}_{\mathbf{e}_2^{-1},\mathbf{e}_1^{-1}}(t,q) &= \sum_{\nu} C_{\bullet\lambda_2^\vee}{}^{\nu^\vee}(t,q) C_{\nu^\vee\lambda_1^\vee}{}^{\bullet}(t,q) Q_{1,2}^{|\nu|} f_{\nu^\vee}(t,q) \\ &= \sum_{\nu} C^{\bullet\lambda_2}{}_{\nu}(q,t) C^{\nu\lambda_1}{}_{\bullet}(q,t) Q_{1,2}^{|\nu|} f_\nu(q,t)^{-1}(-1)^{|\lambda_1|+|\lambda_2|}. \end{aligned} \quad (8.2)$$

Then, from (4.3) and (5.5), it follows that

$$\begin{aligned} \widetilde{Z} &:= \sum_{\lambda_1,\lambda_2} Z^{\lambda_1,\lambda_2}_{\mathbf{e}_1,\mathbf{e}_2}(q,t)\, Z^{\lambda_2^\vee,\lambda_1^\vee}_{\mathbf{e}_2^{-1},\mathbf{e}_1^{-1}}(t,q)\, (\Lambda Q_{1,2})^{|\lambda_1|+|\lambda_2|} f_{\lambda_1}(q,t)/f_{\lambda_2}(q,t) \\ &= \sum_{\lambda_1,\lambda_2} \Pi_0\left(-v^{-1}Q_{1,2}\, q^{\lambda_1}t^\rho,\ t^{\lambda_2^\vee}q^\rho\right)^{-1} \Pi_0\left(-v^{-1}Q_{2,1}\, t^{\lambda_1^\vee}q^\rho,\ q^{\lambda_2}t^\rho\right)^{-1} \\ &\times \prod_{s\in\lambda_2} v^{-1}\Lambda \frac{1}{\left(1-q^{a(s)}t^{\ell(s)+1}\right)\left(1-q^{-a(s)-1}t^{-\ell(s)}\right)} \\ &\times \prod_{s\in\lambda_1} v^{-1}\Lambda \frac{1}{\left(1-q^{a(s)}t^{\ell(s)+1}\right)\left(1-q^{-a(s)-1}t^{-\ell(s)}\right)}. \end{aligned} \quad (8.3)$$

If we factor out the Λ-independent part $\widetilde{Z}^{\mathrm{pert}} := Z^{\bullet,\bullet}_{\mathbf{e}_1,\mathbf{e}_2}(q,t)\, Z^{\bullet,\bullet}_{\mathbf{e}_2^{-1},\mathbf{e}_1^{-1}}(t,q)$, then $Z^{\mathrm{inst}} := \widetilde{Z}/\widetilde{Z}^{\mathrm{pert}}$ agrees with the $SU(2)$ Nekrasov's formula $Z_0^{\mathrm{inst}}\left(\mathbf{e}_1,\mathbf{e}_2,\Lambda^{\frac{1}{4}};q,t\right)$ in (2.4).

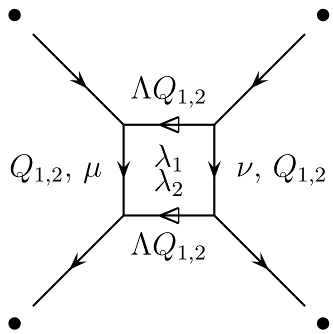


Figure 11: $SU(2)$ partition function: the framing indices for the bottom and the right internal line are -1 and those for the top and the left internal lines are one.

8.2 Pure $SU(N_c)$ partition function

The pure $SU(N_c)$ partition function with Chern-Simons terms is written as follows. Let

$$
\begin{aligned}
&Z^{\lambda_1,\cdots,\lambda_{N_c}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{N_c}}(q,t) \\
&:= \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{N_c} C_{\mu_{\alpha-1}\lambda_\alpha}{}^{\mu_\alpha}(q,t) \prod_{\alpha=1}^{N_c-1} Q^{|\mu_\alpha|}_{\alpha,\alpha+1} f_{\mu_\alpha}(q,t) \\
&= \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{N_c} \sum_{\sigma_\alpha} \iota P_{\mu_{\alpha-1}{}^\vee/\sigma_\alpha{}^\vee}\left(-t^{\lambda_\alpha{}^\vee} q^\rho; t, q\right) P_{\lambda_\alpha}(t^\rho; q, t)\, P_{\mu_\alpha/\sigma_\alpha}\left(q^{\lambda_\alpha} t^\rho; q, t\right) \prod_{\alpha=1}^{N_c-1} v^{|\sigma_\alpha|-|\mu_\alpha|} Q^{|\mu_\alpha|}_{\alpha,\alpha+1},
\end{aligned}
\tag{8.4}
$$

with $Q_{\alpha,\beta} = \mathbf{e}_\alpha/\mathbf{e}_\beta$ and $\mu_0 = \mu_{N_c} = 0$. Note that $\sigma_1 = \sigma_{N_c} = 0$. From the OPE formula (5.10), we have

$$
Z^{\lambda_1,\cdots,\lambda_{N_c}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{N_c}}(q,t) = \prod_{\alpha<\beta} \Pi_0\left(-v^{-1} Q_{\alpha,\beta}\, q^{\lambda_\alpha} t^\rho,\ t^{\lambda_\beta{}^\vee} q^\rho\right)^{-1} \prod_{\alpha=1}^{N_c} P_{\lambda_\alpha}(t^\rho; q, t). \tag{8.5}
$$

The dual part is

$$
Z^{\lambda_{N_c}{}^\vee,\cdots,\lambda_1{}^\vee}_{\mathbf{e}_{N_c}^{-1},\cdots,\mathbf{e}_1^{-1}}(t,q) = \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{N_c} C^{\mu_\alpha\lambda_\alpha}{}_{\mu_{\alpha-1}}(q,t) \prod_{\alpha=1}^{N_c-1} Q^{|\mu_\alpha|}_{\alpha,\alpha+1} f_{\mu_\alpha}(q,t)^{-1} (-1)^{|\lambda_\alpha|}, \tag{8.6}
$$

with $Q_{\alpha,\beta} := \mathbf{e}_\alpha/\mathbf{e}_\beta$. Then, using $\Lambda_{\alpha,m}$ in (2.22),

$$
\widetilde{Z}_m := \sum_{\lambda_1,\cdots,\lambda_{N_c}} Z^{\lambda_1,\cdots,\lambda_{N_c}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{N_c}}(q,t)\, Z^{\lambda_{N_c}{}^\vee,\cdots,\lambda_1{}^\vee}_{\mathbf{e}_{N_c}^{-1},\cdots,\mathbf{e}_1^{-1}}(t,q) \prod_{\alpha=1}^{N_c} \Lambda_{\alpha,m}{}^{|\lambda_\alpha|} f_{\lambda_\alpha}(q,t)^{N_c-m-2\alpha+1}
$$

$$= \sum_{\lambda_1,\cdots,\lambda_{N_c}} \prod_{\alpha<\beta} \Pi_0 \left(-v^{-1}Q_{\alpha,\beta}\, q^{\lambda_\alpha} t^{\rho},\ t^{\lambda_\beta{}^\vee} q^{\rho}\right)^{-1} \Pi_0 \left(-v^{-1}Q_{\beta,\alpha}\, t^{\lambda_\alpha{}^\vee} q^{\rho},\ q^{\lambda_\beta} t^{\rho}\right)^{-1}$$
$$\times \prod_{\alpha=1}^{N_c} f_{\lambda_\alpha}(q,t)^{-m} \prod_{s\in\lambda_\alpha} \frac{v^{-1}\Lambda^{2N_c}\left(-Q_\alpha\right)^{-m}}{\left(1-q^{a(s)}t^{\ell(s)+1}\right)\left(1-q^{-a(s)-1}t^{-\ell(s)}\right)}, \tag{8.7}$$

with $\mu_0 = \mu_{N_c} = \nu_0 = \nu_{N_c} = 0$. If we factor out the Λ-independent part $\widetilde{Z}^{\mathrm{pert}} := Z^{\bullet,\cdots,\bullet}_{\mathbf{e}_1,\cdots,\mathbf{e}_{N_c}}(q,t)\, Z^{\bullet,\cdots,\bullet}_{\mathbf{e}_{N_c}^{-1},\cdots,\mathbf{e}_1^{-1}}(t,q)$, then $Z^{\mathrm{inst}}_m := \widetilde{Z}_m/\widetilde{Z}^{\mathrm{pert}}$ agrees with the $SU(N_c)$ Nekrasov's formula $Z^{\mathrm{inst}}_m(\mathbf{e}_1,\cdots,\mathbf{e}_{N_c},\Lambda;q,t)$ in (2.4).

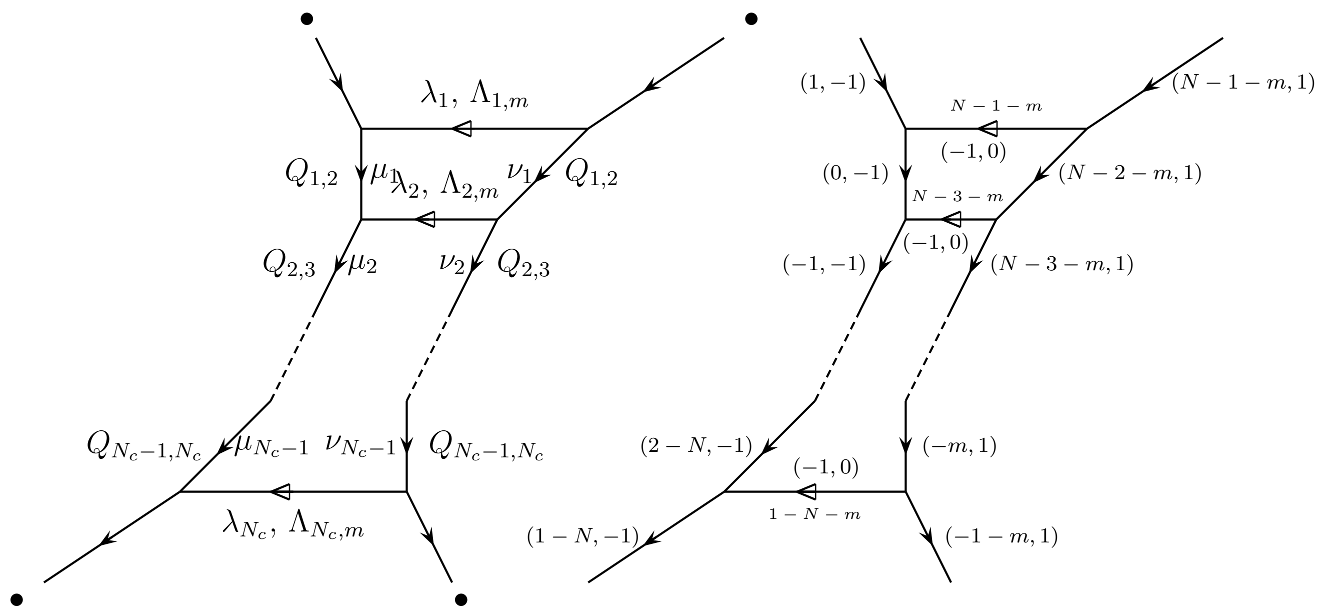


Figure 12: $SU(N_c)$ partition function: the framing indices for the horizontal lines are $N-1-m$, $N-3-m$,$\cdots$,$1-N-m$ from the top to the bottom. Those for the left and the right internal lines are 1 and -1, respectively.

9 $SU(N_c)$ with $N_f = 2N_c$

The partition functions with fundamental matters are also realized by the refined topological vertex as follows. Let

$$Z^{\lambda_1,\cdots,\lambda_{2N_c-1}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{2N_c-1}}(q,t)$$
$$:= \sum_{\{\mu_\alpha\}} \prod_{\alpha=1}^{N_c} C_{\mu_{2\alpha-2}\lambda_{2\alpha-1}}{}^{\mu_{2\alpha-1}}(q,t) C^{\mu_{2\alpha}\lambda_{2\alpha}}{}_{\mu_{2\alpha-1}}(q,t) \prod_{\alpha=1}^{2N_c-1} Q_{\alpha,\alpha+1}^{|\mu_\alpha|}$$

$$=\sum_{\{\mu_\alpha\}}\prod_{\alpha=1}^{N_c}\sum_{\sigma_{2\alpha-1}}\iota P_{\mu_{2\alpha-2}{}^\vee/\sigma_{2\alpha-1}{}^\vee}\left(-t^{\lambda_{2\alpha-1}{}^\vee}q^\rho;t,q\right)P_{\lambda_{2\alpha-1}}\left(t^\rho;q,t\right)P_{\mu_{2\alpha-1}/\sigma_{2\alpha-1}}\left(q^{\lambda_{2\alpha-1}}t^\rho;q,t\right)$$
$$\times\sum_{\sigma_{2\alpha}}P_{\mu_{2\alpha-1}{}^\vee/\sigma_{2\alpha}{}^\vee}\left(-t^{\lambda_{2\alpha}{}^\vee}q^\rho;t,q\right)P_{\lambda_{2\alpha}{}^\vee}\left(-q^\rho;t,q\right)\iota P_{\mu_{2\alpha}/\sigma_{2\alpha}}\left(q^{\lambda_{2\alpha}}t^\rho;q,t\right)$$
$$\times\prod_{\alpha=1}^{N_c}v^{|\sigma_{2\alpha-1}|-|\sigma_{2\alpha}|}\prod_{\alpha=1}^{2N_c-1}Q_{\alpha,\alpha+1}^{|\mu_\alpha|},\tag{9.1}$$

with $\mu_0=\mu_{2N_c}=\sigma_0=\sigma_{2N_c}=0$. As in the pure $SU(N_c)$ case, from (5.10) we have

$$Z^{\lambda_1,\cdots,\lambda_{2N_c-1}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{2N_c-1}}\left(q,t\right)$$
$$=\prod_{\alpha<\beta}\Pi_0\left(-v^{\frac{(-1)^\alpha+(-1)^\beta}{2}}Q_{\alpha,\beta}\,q^{\lambda_\alpha}t^\rho,\ t^{\lambda_\beta{}^\vee}q^\rho\right)^{(-1)^{\alpha+\beta+1}}\prod_{\alpha=1}^{N_c}P_{\lambda_{2\alpha-1}}\left(t^\rho;q,t\right)P_{\lambda_{2\alpha}{}^\vee}\left(-q^\rho;t,q\right)$$
$$=\prod_{\alpha<\beta}\frac{\Pi_0\left(-Q_{2\alpha,2\beta-1}\,q^{\lambda_{2\alpha}}t^\rho,\ t^{\lambda_{2\beta-1}{}^\vee}q^\rho\right)\Pi_0\left(-Q_{2\alpha-1,2\beta}\,q^{\lambda_{2\alpha-1}}t^\rho,\ t^{\lambda_{2\beta}{}^\vee}q^\rho\right)}{\Pi_0\left(-vQ_{2\alpha,2\beta}\,q^{\lambda_{2\alpha}}t^\rho,\ t^{\lambda_{2\beta}{}^\vee}q^\rho\right)\Pi_0\left(-v^{-1}Q_{2\alpha-1,2\beta-1}\,q^{\lambda_{2\alpha-1}}t^\rho,\ t^{\lambda_{2\beta-1}{}^\vee}q^\rho\right)}$$
$$\times\prod_{\alpha=1}^{N_c}P_{\lambda_{2\alpha-1}}\left(t^\rho;q,t\right)P_{\lambda_{2\alpha}{}^\vee}\left(-q^\rho;t,q\right).\tag{9.2}$$

The dual part is

$$Z^{\lambda_{2N_c-1}{}^\vee,\cdots,\lambda_1{}^\vee}_{\mathbf{e}'^{-1}_{2N_c-1},\cdots,\mathbf{e}'^{-1}_1}\left(t,q\right)=\sum_{\{\nu_\alpha\}}\prod_{\alpha=1}^{N_c}C^{\nu_{2\alpha-1}\lambda_{2\alpha-1}}{}_{\nu_{2\alpha-2}}(q,t)C_{\nu_{2\alpha-1}\lambda_{2\alpha}}{}^{\nu_{2\alpha}}(q,t)\prod_{\alpha=1}^{2N_c-1}Q'^{|\nu_\alpha|}_{\alpha,\alpha+1}(-1)^{|\lambda_{2\alpha-1}|},\tag{9.3}$$

with $Q'_{\alpha,\beta}=\mathbf{e}'_\alpha/\mathbf{e}'_\beta$ and $\mathbf{e}'_{2\alpha-1}=\mathbf{e}_{2\alpha-1}$.

When $\lambda_{2\alpha}$ for even integers 2α is a trivial representation, let

$$\widetilde{Z}:=\sum_{\lambda_1,\lambda_3,\cdots,\lambda_{2N_c-1}}Z^{\lambda_1,\bullet,\lambda_3,\cdots,\bullet,\lambda_{2N_c-1}}_{\mathbf{e}_1,\cdots,\mathbf{e}_{2N_c-1}}\left(q,t\right)Z^{\lambda_{2N_c-1}{}^\vee,\bullet,\cdots,\lambda_3{}^\vee,\bullet,\lambda_1{}^\vee}_{\mathbf{e}'^{-1}_{2N_c-1},\cdots,\mathbf{e}'^{-1}_1}\left(t,q\right)\prod_{\alpha=1}^{N_c}\Lambda_\alpha^{|\lambda_{2\alpha-1}|}f_{\lambda_{2\alpha-1}}\left(q,t\right)^{-1},$$
$$\Lambda_\alpha:=v^{-1}\Lambda^{2N_c}\prod_{\beta=1}^{\alpha-1}\frac{\mathbf{e}_{2\beta-1}}{\mathbf{e}'_{2\beta}}\prod_{\beta=\alpha}^{N_c}\frac{\mathbf{e}'_{2\beta}}{\mathbf{e}_{2\beta-1}}.\tag{9.4}$$

In addition, let $Z^{\mathrm{inst}}:=\widetilde{Z}/\widetilde{Z}^{\mathrm{pert}}$ with $\widetilde{Z}^{\mathrm{pert}}:=Z^{\bullet,\cdots,\bullet}_{\mathbf{e}_1,\cdots,\mathbf{e}_{2N_c-1}}\left(q,t\right)Z^{\bullet,\cdots,\bullet}_{\mathbf{e}'^{-1}_{2N_c-1},\cdots,\mathbf{e}'^{-1}_1}\left(t,q\right)$. Then

$$Z^{\mathrm{inst}}=\sum_{\{\lambda_{2\alpha-1}\}}\frac{\prod_{\alpha=1}^{N_c}\Lambda_\alpha^{|\lambda_{2\alpha-1}|}f_{\lambda_{2\alpha-1}}\left(q,t\right)^{-1}}{\prod_{\alpha<\beta}\left(N_{\lambda_\alpha\lambda_\beta}\left(Q_{\alpha,\beta};q,t\right)N_{\lambda_\beta{}^\vee\lambda_\alpha{}^\vee}\left(Q'_{\alpha,\beta};t,q\right)\right)^{(-1)^{\alpha+\beta}}\prod_{\alpha=1}^{N_c}N_{\lambda_{2\alpha-1}\lambda_{2\alpha-1}}\left(1;q,t\right)}$$
$$=\sum_{\{\lambda_{2\alpha-1}\}}\frac{\prod_{\alpha=1}^{N_c}\Lambda^{2N_c|\lambda_{2\alpha-1}|}}{\prod_{\alpha<\beta}\left(N_{\lambda_\alpha\lambda_\beta}\left(Q_{\alpha,\beta};q,t\right)N_{\lambda_\beta\lambda_\alpha}\left(Q'_{\beta,\alpha};q,t\right)\right)^{(-1)^{\alpha+\beta}}\prod_{\alpha=1}^{N_c}N_{\lambda_{2\alpha-1}\lambda_{2\alpha-1}}\left(1;q,t\right)},\tag{9.5}$$

gives the $SU(N_c)$ partition function with $N_f=2N_c$.

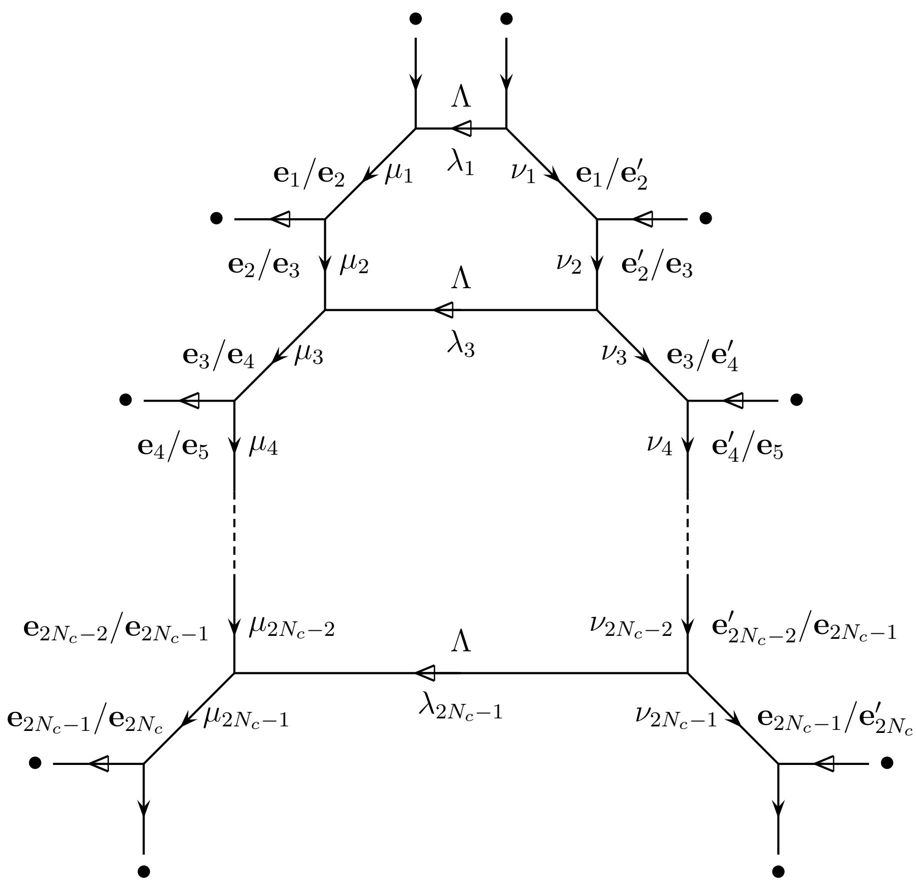


Figure 13: $SU(N_c)$ partition function with $N_f = 2N_c$: the framing index for the longitudinal internal lines are all -1.

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Appendix A : Proof of the Proposition in Sect. 2.1

A.1 Combinatorial identities

We have the following formula for the Young diagrams, which translates the summation in squares into that in lows:

Lemma. For all integers $N_\lambda \geq \ell(\lambda)$ and $N_\mu \geq \ell(\mu)$,

$$(1-q)\sum_{(i,j)\in\lambda} q^{j-1}t^{-i+1} = \sum_{i=1}^{N_\lambda}\left(1-q^{\lambda_i}\right)t^{-i+1}, \tag{A.1}$$

$$(1-q)\sum_{(i,j)\in\mu} q^{\lambda_i-j}t^{\mu_j^\vee-i} = \left(\sum_{i=1}^{N_\mu}\sum_{j=i}^{N_\mu} - t^{-1}\sum_{i=1}^{N_\mu}\sum_{j=i+1}^{N_\mu+1}\right) q^{\lambda_i-\mu_j}t^{j-i}. \tag{A.2}$$

Proof. (A.1) follows from $\sum_{j=1}^{\lambda} q^{j-1} = (1-q^\lambda)/(1-q)$.

The left-hand side of (A.2) reduces to

$$\begin{aligned}(1-q)\sum_{i=1}^{\ell(\mu)}\sum_{k=0}^{\ell(\mu)-i} t^k q^{\lambda_i-\mu_{i+k}}\sum_{\ell=0}^{\mu_{i+k}-\mu_{i+k+1}-1} q^\ell &= \sum_{i=1}^{\ell(\mu)}\sum_{k=0}^{\ell(\mu)-i} t^k\left(q^{\lambda_i-\mu_{i+k}} - q^{\lambda_i-\mu_{i+k+1}}\right)\\ &= \sum_{i=1}^{N_\mu}\sum_{j=i}^{N_\mu} t^{j-i}\left(q^{\lambda_i-\mu_j} - q^{\lambda_i-\mu_{j+1}}\right),\end{aligned} \tag{A.3}$$

which equals the right-hand side of (A.2). ∎

From

$$\sum_{1\leq i<j\leq N+1} q^{\lambda_i-\mu_j}t^{j-i} = \sum_{1\leq i<j\leq N} q^{\lambda_i-\mu_j}t^{j-i} + \sum_{1\leq i\leq N} q^{\lambda_i-\mu_{N+1}}t^{N+1-i}, \tag{A.4}$$

(A.2) is rewritten as

$$(1-q)\sum_{(i,j)\in\mu} q^{\lambda_i-j}t^{\mu_j^\vee-i+1} = (t-1)\sum_{1\leq i<j\leq N_\mu} q^{\lambda_i-\mu_j}t^{j-i} + t\sum_{i=1}^{N_\mu} q^{\lambda_i}\left(q^{-\mu_i} - t^{N-i}\right). \tag{A.5}$$

Note that if $t=q$ and $\lambda=\mu$, (A.5) reduces to the formula of the Maya diagram: the length from a black box to a white one or black one is $(\lambda_i - i) + (\lambda_j^\vee - j) + 1$ (the hook length) or $(\lambda_i - i) - (\lambda_j - j)$, respectively:

$$\sum_{(i,j)\in\lambda} q^{(\lambda_i-i)+(\lambda_j^\vee-j)+1} + \sum_{1\leq i<j\leq N_\lambda} q^{(\lambda_i-i)-(\lambda_j-j)} = \sum_{1\leq i\leq N_\lambda}\ \sum_{i<j\leq\lambda_i+N_\lambda} q^{j-i}. \tag{A.6}$$

By using (A.1), we have:
Lemma. For all integers $N_\lambda \geq \ell(\lambda)$ and $N_{\lambda^\vee} \geq \ell(\lambda^\vee)$,

$$\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \sum_{i=1}^{N_\lambda} \left(q^{\lambda_i} - 1\right) t^{\frac{1}{2}-i} + \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) \sum_{i=1}^{N_{\lambda^\vee}} \left(t^{-\lambda_i^\vee} - 1\right) q^{i-\frac{1}{2}} = 0. \tag{A.7}$$

Proof. Similar to (A.1), for all integers $N_{\lambda^\vee} \geq \ell(\lambda^\vee)$,

$$\sum_{i=1}^{N_{\lambda^\vee}} \left(1 - t^{-\lambda_i^\vee}\right) q^{i-1} = (1 - t^{-1}) \sum_{(i,j)\in\lambda^\vee} t^{1-j} q^{i-1} = (1 - t^{-1}) \sum_{(i,j)\in\lambda} t^{1-i} q^{j-1}. \tag{A.8}$$

Therefore, with (A.1),

$$(1 - q) \sum_{i=1}^{N_{\lambda^\vee}} \left(1 - t^{-\lambda_i^\vee}\right) q^{i-1} = (1 - t^{-1}) \sum_{i=1}^{N_\lambda} \left(1 - q^{\lambda_i}\right) t^{1-i}. \tag{A.9}$$

□

In the power sum function (1.8), (A.7) is written as

$$\left(t^{\frac{n}{2}} - t^{-\frac{n}{2}}\right) p_n \left(q^\lambda t^\rho, L t^{-\rho}\right) + \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right) p_n \left(t^{-\lambda^\vee} q^{-\rho}, L q^\rho\right) = 0, \qquad L \in \mathbb{C}. \tag{A.10}$$

Note that if $t = q$, (A.7) reduces to the formula of the Maya diagram: the black boxes and the white ones are at $\lambda_i - i + \frac{1}{2}$ and $-(\lambda_i^\vee - i + \frac{1}{2})$ of the Maya diagram, respectively:

$$\sum_{i=1}^{N_\lambda} q^{\lambda_i - i + \frac{1}{2}} + \sum_{i=1}^{N_{\mu^\vee}} q^{-\lambda_i^\vee + i - \frac{1}{2}} = \sum_{i=1-N_\lambda}^{N_{\mu^\vee}} q^{i-\frac{1}{2}}. \tag{A.11}$$

Hence $\sum_{i\geq 1} q^{\lambda_i - i + \frac{1}{2}} + \sum_{i\geq 1} q^{-\lambda_i^\vee + i - \frac{1}{2}} = \sum_{i\in\mathbb{Z}} q^{i-\frac{1}{2}} = q^{-\frac{1}{2}} \delta(q)$.

A.2 Factors in Nekrasov's formula

We have the following formula for the Young diagrams, which implies the equivalence among several expressions of Nekrasov's formula.
Proposition. The following $n^{i,\pm}_{\lambda\mu}(L_1, L_2; q, t)$'s $(i = 1, 2, 3)$ are all the same.

$$vn^{1,+}_{\lambda\mu}(L_1, L_2; q, t) := \sum_{(i,j)\in\mu} \left(q^{\lambda_i} - L_1\right) q^{\frac{1}{2}-j} t^{\mu_j^\vee - i + \frac{1}{2}} + \sum_{(i,j)\in\lambda} \left(q^{-\mu_i} - L_2\right) q^{j-\frac{1}{2}} t^{-\lambda_j^\vee + i - \frac{1}{2}},$$
$$vn^{1,-}_{\lambda\mu}(L_1, L_2; q, t) := \sum_{(i,j)\in\lambda} q^{\lambda_i - j + \frac{1}{2}} t^{\frac{1}{2}-i} \left(t^{\mu_j^\vee} - L_2\right) + \sum_{(i,j)\in\mu} q^{-\mu_i + j - \frac{1}{2}} t^{i-\frac{1}{2}} \left(t^{-\lambda_j^\vee} - L_1\right), \tag{A.12}$$

$$vn^{2,+}_{\lambda\mu}(L_1,L_2;q,t) := p_1\left(q^{\lambda}t^{\rho},L_1t^{-\rho}\right)p_1\left(t^{\mu^\vee}q^{\rho},L_2q^{-\rho}\right) - p_1\left(t^{\rho},L_1t^{-\rho}\right)p_1\left(q^{\rho},L_2q^{-\rho}\right),$$
$$vn^{2,-}_{\lambda\mu}(L_1,L_2;q,t) := p_1\left(t^{-\lambda^\vee}q^{-\rho},L_1q^{\rho}\right)p_1\left(q^{-\mu}t^{-\rho},L_2t^{\rho}\right) - [\ \lambda=\mu=0\ ], \tag{A.13}$$

$$vn^{3,+}_{\lambda\mu}(L_1,L_2;q,t) := \left\{p_1\left(q^{\lambda}t^{\rho},\ L_1t^{-\rho}\right)p_1\left(q^{-\mu}t^{-\rho},\ L_2t^{\rho}\right) - [\ \lambda=\mu=0\ ]\right\}\frac{t^{-\frac{1}{2}}-t^{\frac{1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},$$
$$vn^{3,-}_{\lambda\mu}(L_1,L_2;q,t) := \left\{p_1\left(t^{-\lambda^\vee}q^{-\rho},\ L_1q^{\rho}\right)p_1\left(t^{\mu^\vee}q^{\rho},\ L_2q^{-\rho}\right) - [\ \lambda=\mu=0\ ]\right\}\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{t^{-\frac{1}{2}}-t^{\frac{1}{2}}}, \tag{A.14}$$

with $v := (q/t)^{\frac{1}{2}}$ and $L_1,\ L_2 \in \mathbb{C}$. Here p_1 is the power sum function in (1.8) and $[\ \lambda=\mu=0\ ]$'s stand for terms substituting $\lambda=\mu=0$ into the foregoing ones.

Proof. It is clear that

$$\begin{aligned}
n^{1,\pm}_{\lambda\mu}(L_1,L_2;q,t)v &= n^{1,\pm}_{\mu\lambda}(L_2,L_1;q^{-1},t^{-1})/v = n^{1,\mp}_{\mu^\vee\lambda^\vee}(L_2,L_1;t,q)/v,\\
n^{2,\pm}_{\lambda\mu}(L_1,L_2;q,t)v &= n^{2,\mp}_{\mu\lambda}(L_2,L_1;q^{-1},t^{-1})/v = n^{2,\pm}_{\mu^\vee\lambda^\vee}(L_2,L_1;t,q)/v,\\
n^{3,\pm}_{\lambda\mu}(L_1,L_2;q,t)v &= n^{3,\pm}_{\mu\lambda}(L_2,L_1;q^{-1},t^{-1})/v = n^{3,\mp}_{\mu^\vee\lambda^\vee}(L_2,L_1;t,q)/v.
\end{aligned} \tag{A.15}$$

Therefore, it suffices to show $n^{1+}_{\lambda\mu} = n^{2+}_{\lambda\mu} = n^{3+}_{\lambda\mu}$. First, applying (A.10) yields $n^{2+}_{\lambda\mu} = n^{3+}_{\lambda\mu}$. Next, we prove that $n^{1+}_{\lambda\mu} = n^{3+}_{\lambda\mu}$. From (A.2)$\times t$, we have, for all integers $N_{\lambda\mu} \geq \ell(\lambda)$, $\ell(\mu)$,

$$\sum_{(i,j)\in\mu} q^{\lambda_i-j}t^{\mu^\vee_j-i+1} = \frac{1}{1-q}\left[\ t\sum_{i=1}^{N_{\lambda\mu}}\sum_{j=i}^{N_{\lambda\mu}} - \sum_{i=1}^{N_{\lambda\mu}}\sum_{j=i+1}^{N_{\lambda\mu}+1}\ \right] q^{\lambda_i-\mu_j}t^{j-i}. \tag{A.16}$$

By replacing q, t and λ in (A.2) with $1/q$, $1/t$ and μ, respectively,

$$\sum_{(i,j)\in\lambda} q^{-\mu_i+j-1}t^{-\lambda^\vee_j+i} = \frac{1}{1-q}\left[\ t\sum_{j=1}^{N_{\lambda\mu}}\sum_{i=j+1}^{N_{\lambda\mu}+1} - \sum_{j=1}^{N_{\lambda\mu}}\sum_{i=j}^{N_{\lambda\mu}}\ \right] q^{\lambda_i-\mu_j}t^{j-i}. \tag{A.17}$$

Adding these two equations, we have

$$\begin{aligned}
n^{1,+}_{\lambda\mu}&(L_1,L_2;q,t) + L_1\sum_{(i,j)\in\mu} q^{-j}t^{i} + L_2\sum_{(i,j)\in\lambda} q^{j-1}t^{1-i}\\
&= \sum_{(i,j)\in\mu} q^{\lambda_i-j}t^{\mu^\vee_j-i+1} + \sum_{(i,j)\in\lambda} q^{-\mu_i+j-1}t^{-\lambda^\vee_j+i}\\
&= \frac{1}{1-q}\left[\ t\sum_{i=1}^{N_{\lambda\mu}+1}\sum_{j=1}^{N_{\lambda\mu}} - \sum_{i=1}^{N_{\lambda\mu}}\sum_{j=1}^{N_{\lambda\mu}+1}\ \right] q^{\lambda_i-\mu_j}t^{j-i}\\
&= \frac{1}{1-q}\left[\ t\sum_{i=1}^{N_{\lambda\mu}+1}\sum_{j=1}^{N_{\lambda\mu}} - \sum_{i=1}^{N_{\lambda\mu}}\sum_{j=1}^{N_{\lambda\mu}+1}\ \right] \left(q^{\lambda_i-\mu_j}-1\right)t^{j-i}.
\end{aligned} \tag{A.18}$$

Thus, the following lemma with $N_\lambda = N_\mu = N_{\lambda\mu}$ shows that $n^{1+}_{\lambda\mu} = n^{3+}_{\lambda\mu}$. □

Lemma. For any integers $N_\lambda \geq \ell(\lambda)$, $N_\mu \geq \ell(\mu)$ and $N_{\mu^\vee} \geq \ell(\mu^\vee)$,

$$\begin{aligned}
n^{2,+}_{\lambda\mu}(L_1, L_2; q, t) &= \sum_{i=1}^{N_\lambda} \sum_{j=1}^{N_{\mu^\vee}} \left(q^{\lambda_i} t^{\mu^\vee_j} - 1 \right) t^{1-i} q^{-j} \\
&\quad + \sum_{(i,j)\in\lambda} q^{\lambda_i - j} t^{1-i} \left(q^{-N_{\mu^\vee}} - L_2 \right) + \sum_{(i,j)\in\mu} t^{\mu^\vee_j - i + 1} q^{-j} \left(t^{-N_\lambda} - L_1 \right), \\
n^{3,+}_{\lambda\mu}(L_1, L_2; q, t) &= \frac{1}{1-q} \left[t \sum_{i=1}^{N_\lambda+1} \sum_{j=1}^{N_\mu} - \sum_{i=1}^{N_\lambda} \sum_{j=1}^{N_\mu+1} \right] \left(q^{\lambda_i - \mu_j} - 1 \right) t^{j-i} \\
&\quad - L_1 \sum_{(i,j)\in\mu} q^{-j} t^i - L_2 \sum_{(i,j)\in\lambda} q^{j-1} t^{1-i}.
\end{aligned} \tag{A.19}$$

Proof.

$$\begin{aligned}
&t^{-1} n^{2,+}_{\lambda\mu}(L_1, L_2; q, t) \\
&= \left(\sum_{i=1}^{N_\lambda} \left(q^{\lambda_i} - 1 \right) t^{-i} - \frac{1-L_1}{1-t} \right) \left(\sum_{j=1}^{N_{\mu^\vee}} \left(t^{\mu^\vee_j} - 1 \right) q^{-j} - \frac{1-L_2}{1-q} \right) - \frac{1-L_1}{1-t} \frac{1-L_2}{1-q} \\
&= \left(\sum_{i=1}^{N_\lambda} q^{\lambda_i} t^{-i} - \frac{t^{-N_\lambda} - L_1}{1-t} \right) \left(\sum_{j=1}^{N_{\mu^\vee}} t^{\mu^\vee_j} q^{-j} - \frac{q^{-N_{\mu^\vee}} - L_2}{1-q} \right) - [\ \lambda = \mu = 0\ ] \\
&= \sum_{i=1}^{N_\lambda} \sum_{j=1}^{N_{\mu^\vee}} \left(q^{\lambda_i} t^{\mu^\vee_j} - 1 \right) t^{1-i} q^{-j} + \sum_{i=1}^{N_\lambda} \frac{1-q^{\lambda_i}}{1-q} t^{1-i} \left(q^{-N_{\mu^\vee}} - L_2 \right) + \sum_{j=1}^{N_{\mu^\vee}} \frac{t^{\mu^\vee_j - 1}}{1-t^{-1}} q^{-j} \left(t^{-N_\lambda} - L_1 \right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1-q}{t-1} n^{3,+}_{\lambda\mu}(L_1, L_2; q, t) \\
&= \left(\sum_{i=1}^{N_\lambda} \left(q^{\lambda_i} - 1 \right) t^{-i} - \frac{1-L_1}{1-t} \right) \left(\sum_{j=1}^{N_\mu} \left(q^{-\mu_j} - 1 \right) t^j - \frac{1-L_2}{1-t^{-1}} \right) - \frac{1-L_1}{1-t} \frac{1-L_2}{1-t^{-1}} \\
&= \left(\sum_{i=1}^{N_\lambda} q^{\lambda_i} t^{-i} - \frac{t^{-N_\lambda} - L_1}{1-t} \right) \left(\sum_{j=1}^{N_\mu} q^{-\mu_j} t^j - \frac{t^{N_\mu} - L_2}{1-t^{-1}} \right) - [\ \lambda = \mu = 0\ ] \\
&= \sum_{i=1}^{N_\lambda} \sum_{j=1}^{N_\mu} \left(q^{\lambda_i - \mu_j} - 1 \right) t^{j-i} - \sum_{j=1}^{N_\mu} \left(q^{-\mu_j} - 1 \right) t^j \frac{t^{-N_\lambda} - L_1}{1-t} - \sum_{i=1}^{N_\lambda} \left(q^{\lambda_i} - 1 \right) t^{-i} \frac{t^{N_\mu} - L_2}{1-t^{-1}}.
\end{aligned}$$

□

Note that $n^{2\pm}_{\lambda\mu}$ and $n^{3\pm}_{\lambda\mu}$ are independent of N_λ's, if they are sufficiently large. Let $n_{\lambda\mu}(L_1, L_2; q, t) := n^{i,\pm}_{\lambda\mu}(L_1, L_2; q, t)$; then it satisfies

$$n_{\lambda\mu}(L_1, L_2; q, t) v = n_{\mu\lambda}(L_2, L_1; q^{-1}, t^{-1})/v = n_{\mu^\vee\lambda^\vee}(L_2, L_1; t, q)/v. \tag{A.20}$$

Let

$$N_{\lambda\mu}^{i,\pm}(vQ,L_1,L_2;q,t) := \exp\left\{-\sum_{n>0}\frac{Q^n}{n}v^n n_{\lambda\mu}^{i,\pm}(L_1^n,L_2^n;q^n,t^n)\right\}, \tag{A.21}$$

then we have:

Corollary. The following $N_{\lambda\mu}^{i,\pm}(Q,L_1,L_2;q,t)$'s $(i=1,2,3)$ are all the same.

$$\begin{aligned}
N_{\lambda\mu}^{1,+}(vQ,L_1,L_2;q,t) &= \prod_{(i,j)\in\mu}\frac{1-Q\,q^{\lambda_i-j+\frac12}t^{\mu_j^\vee-i+\frac12}}{1-QL_1\,q^{\frac12-j}t^{i-\frac12}}\prod_{(i,j)\in\lambda}\frac{1-Q\,q^{-\mu_i+j-\frac12}t^{-\lambda_j^\vee+i-\frac12}}{1-QL_2\,q^{j-\frac12}t^{\frac12-i}},\\
N_{\lambda\mu}^{1,-}(vQ,L_1,L_2;q,t) &= \prod_{(i,j)\in\lambda}\frac{1-Q\,q^{\lambda_i-j+\frac12}t^{\mu_j^\vee-i+\frac12}}{1-QL_2\,q^{j-\frac12}t^{\frac12-i}}\prod_{(i,j)\in\mu}\frac{1-Q\,q^{-\mu_i+j-\frac12}t^{-\lambda_j^\vee+i-\frac12}}{1-QL_1\,q^{\frac12-j}t^{i-\frac12}},
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
N_{\lambda\mu}^{2,+}(vQ,L_1,L_2;q,t) &= \frac{\Pi_0\left(-Q\ \left\{q^\lambda t^\rho, L_1t^{-\rho}\right\},\ \left\{t^{\mu^\vee}q^\rho, L_2q^{-\rho}\right\}\right)}{\Pi_0\left(-Q\ \left\{t^\rho, L_1t^{-\rho}\right\},\ \left\{q^\rho, L_2q^{-\rho}\right\}\right)},\\
N_{\lambda\mu}^{2,-}(vQ,L_1,L_2;q,t) &= \frac{\Pi_0\left(-Q\ \left\{t^{-\lambda^\vee}q^{-\rho}, L_1q^\rho\right\},\ \left\{q^{-\mu}t^{-\rho}, L_2t^\rho\right\}\right)}{\Pi_0\left(-Q\ \left\{q^{-\rho}, L_1q^\rho\right\},\ \left\{t^{-\rho}, L_2t^\rho\right\}\right)},
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
N_{\lambda\mu}^{3,+}(vQ,L_1,L_2;q,t) &= \frac{\Pi\left(vQ\ \left\{q^\lambda t^\rho, L_1t^{-\rho}\right\},\ \left\{q^{-\mu}t^{-\rho}, L_2t^\rho\right\};q,t\right)}{\Pi\left(vQ\ \left\{t^\rho, L_1t^{-\rho}\right\},\ \left\{t^{-\rho}, L_2t^\rho\right\};q,t\right)},\\
N_{\lambda\mu}^{3,-}(vQ,L_1,L_2;q,t) &= \frac{\Pi\left(vQ\ \left\{t^{-\lambda^\vee}q^{-\rho}, L_1q^\rho\right\},\ \left\{t^{\mu^\vee}q^\rho, L_2q^{-\rho}\right\};t^{-1},q^{-1}\right)}{\Pi\left(vQ\ \left\{q^{-\rho}, L_1q^\rho\right\},\ \left\{q^\rho, L_2q^{-\rho}\right\};t^{-1},q^{-1}\right)},
\end{aligned} \tag{A.24}$$

By letting L_1 and $L_2=0$, we obtain six expressions of $N_{\lambda\mu}(Q;q,t)$ in Nekrasov's formula. This completes the proof of the proposition in section 2.1.

Appendix B : Formula for the Macdonald Symmetric Function

Here we recapitulate basic properties of the Macdonald symmetric function [15].

B.1 Definition for the Macdonald symmetric function

Bases of the ring of symmetric functions in an infinite number of variables $x=(x_1,x_2,\cdots)$ are indexed by the Young diagram, i.e. the partition $\lambda=(\lambda_1,\lambda_2,\cdots)$, which is a sequence of nonnegative integers such that $\lambda_i\geq\lambda_{i+1}$ and $|\lambda|=\sum_i\lambda_i<\infty$. For example,

the monomial symmetric function is defined by $m_\lambda(x) = \sum_\sigma x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \cdots$, where the summation is over all distinct permutations of $(\lambda_1, \lambda_2, \cdots)$. The power sum symmetric function $p_\lambda(x)$ is defined by

$$p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots, \qquad p_n(x) = \sum_{i=1}^{\infty} x_i^n. \tag{B.1}$$

We introduce an inner product on the ring of symmetric functions in the following manner: for any symmetric functions f and g, in power sums p_λ's,

$$\langle f(p) | g(p) \rangle_{q,t} := f(p^*)\, g(p)\, |_{\text{constant part}}, \qquad p_n^* := n \frac{1-q^n}{1-t^n} \frac{\partial}{\partial p_n}. \tag{B.2}$$

The Macdonald symmetric function $P_\lambda = P_\lambda(x; q, t)$ is uniquely specified by the following orthogonality and normalization:

$$\langle P_\lambda | P_\mu \rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu, \tag{B.3}$$

$$P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu(x), \quad u_{\lambda\mu} \in \mathbb{Q}(q, t). \tag{B.4}$$

Here we used the dominance partial ordering on the Young diagrams defined as $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

The scalar product is given by

$$\langle P_\lambda | P_\lambda \rangle_{q,t} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{\ell(s)}}{1 - q^{a(s)} t^{\ell(s)+1}}, \tag{B.5}$$

which satisfies

$$\langle P_\lambda | P_\lambda \rangle_{q,t} = \left(\frac{q}{t}\right)^{|\lambda|} \langle P_\lambda | P_\lambda \rangle_{q^{-1}, t^{-1}} = \langle P_{\lambda^\vee} | P_{\lambda^\vee} \rangle_{t,q}^{-1}. \tag{B.6}$$

If we define

$$g_\lambda(q, t) := \frac{v^{|\lambda|}}{\langle P_\lambda | P_\lambda \rangle_{q,t}}, \tag{B.7}$$

with $v = (q/t)^{\frac{1}{2}}$, then

$$g_\lambda(q, t) = g_\lambda(q^{-1}, t^{-1}) = g_{\lambda^\vee}(t, q)^{-1}. \tag{B.8}$$

The skew Macdonald symmetric function $P_{\lambda/\mu}(x; q, t)$ is defined by

$$P_{\lambda/\mu}(x; q, t) := g_\mu(q, t) P_\mu^* \left(v^{-1} x; q, t\right)\, P_\lambda(x; q, t), \tag{B.9}$$

where $*$ acts on the power sum as $p_n^* := n \frac{1-q^n}{1-t^n} \frac{\partial}{\partial p_n}$. Finally let $\iota P_{\lambda/\mu}(x; q, t)$ be the skew Macdonald function with the involution ι acting on the power sum p_n as $\iota(p_n) = -p_n$.

Let $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ be two sets of variables. Then we have

$$\sum_{\mu} P_{\lambda/\mu}(x;q,t) P_{\mu/\nu}(y;q,t) = P_{\lambda/\nu}(x,y;q,t), \tag{B.10}$$

where $P_{\lambda/\nu}(x,y;q,t)$ denotes the skew Macdonald function in the set of variables $(x_1, x_2, \cdots, y_1, y_2, \cdots)$.

B.2 Symmetries and Cauchy formulas

Next, we turn to showing the basic properties of the (skew) Macdonald symmetric function. The Macdonald function enjoys the symmetries

$$P_{\lambda/\mu}(cx;q,t) = c^{|\lambda|-|\mu|} P_{\lambda/\mu}(x;q,t), \qquad c \in \mathbb{C}, \tag{B.11}$$

$$P_{\lambda/\mu}\left(x;q^{-1},t^{-1}\right) = P_{\lambda/\mu}(x;q,t), \tag{B.12}$$

$$P_{\lambda^\vee/\mu^\vee}(vx;t,q) = \frac{g_\lambda(q,t)}{g_\mu(q,t)} \omega_{q,t} P_{\lambda/\mu}(x;q,t), \qquad \omega_{q,t}(p_n) = (-1)^{n-1} \frac{1-q^n}{1-t^n} p_n. \tag{B.13}$$

When $t = q$, the Schur function has the extra symmetries

$$s_{\lambda^\vee}(x) = \iota s_\lambda(-x) = (-1)^{|\lambda|} \iota s_\lambda(x). \tag{B.14}$$

The following Cauchy formula is especially important:

$$\begin{aligned} \sum_\lambda g_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) = \Pi(vx,y) &:= \exp\left\{ \sum_{n>0} \frac{v^n}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y) \right\} \\ &= \prod_{k\geq 0} \prod_{i,j} \frac{1-tvx_iy_jq^k}{1-vx_iy_jq^k}, \quad |q|<1. \end{aligned} \tag{B.15}$$

$$\begin{aligned} \sum_\lambda P_\lambda(x;q,t) P_{\lambda^\vee}(y;t,q) = \Pi_0(x,y) &:= \exp\left\{ \sum_{n>0} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y) \right\} \\ &= \prod_{i,j} (1 + x_i y_j). \end{aligned} \tag{B.16}$$

The Cauchy formulas for the skew Macdonald function are

$$\begin{aligned} \sum_\lambda \frac{g_\lambda(q,t)}{g_\mu(q,t)} P_{\lambda/\mu}(x;q,t) P_{\lambda/\nu}(y;q,t) &= \Pi(vx,y) \sum_\lambda P_{\mu/\lambda}(y;q,t) P_{\nu/\lambda}(x;q,t) \frac{g_\nu(q,t)}{g_\lambda(q,t)}, \\ \sum_\lambda P_{\lambda/\mu}(x;q,t) P_{\lambda^\vee/\nu^\vee}(y;t,q) &= \Pi_0(x,y) \sum_\lambda P_{\mu^\vee/\lambda^\vee}(y;t,q) P_{\nu/\lambda}(x;q,t). \end{aligned} \tag{B.17}$$

If we denote by $\omega^x_{q,t}$ the endmorphism $\omega_{q,t}$ on variables x, then

$$\Pi(vx,y;q,t) = \Pi(v^{-1}x,y;q^{-1},t^{-1}) = \omega^x_{t,q} \omega^y_{t,q} \Pi(v^{-1}x,y;t,q). \tag{B.18}$$

B.3 Specialization formulas

We denote

$$
\begin{aligned}
p_n(cq^\lambda t^\rho) &:= c^n \sum_{i=1}^{\infty}(q^{n\lambda_i}-1)t^{n(\frac{1}{2}-i)} + \frac{c^n}{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}, \qquad c\in\mathbb{C},\\
p_n(cq^\lambda t^\rho, cLt^{-\rho}) &:= p_n(cq^\lambda t^\rho) + p_n(cLt^{-\rho}),\\
&= c^n \sum_{i=1}^{\infty}(q^{n\lambda_i}-1)t^{n(\frac{1}{2}-i)} + c^n\frac{1-L^n}{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}, \qquad c, L\in\mathbb{C}. \qquad \text{(B.19)}
\end{aligned}
$$

Then by using (B.13) and (A.10), we obtain

$$
P_{\mu^\vee/\nu^\vee}\left(-t^{\lambda^\vee}q^\rho,\ -Lq^{-\rho};t,q\right) = \frac{g_\mu(q,t)}{g_\nu(q,t)}P_{\mu/\nu}\left(q^{-\lambda}t^{-\rho},\ Lt^\rho;q,t\right), \qquad \text{(B.20)}
$$

The Macdonald function in the power sums $p_n = (1-L^n)/(t^{\frac{n}{2}}-t^{-\frac{n}{2}})$ is [15](Ch. VI.6)

$$
P_\lambda\left(t^\rho, Lt^{-\rho};q,t\right) = \prod_{s\in\lambda}(-1)t^{\frac{1}{2}}q^{a'(s)}\frac{1-Lq^{-a'(s)}t^{\ell'(s)}}{1-q^{a(s)}t^{\ell(s)+1}}, \qquad \text{(B.21)}
$$

for a generic $L\in\mathbb{C}$. By replacing (q,t) and λ with (t,q) and $\lambda^\vee$, respectively,

$$
P_{\lambda^\vee}\left(q^\rho, Lq^{-\rho};t,q\right) = \prod_{s\in\lambda}q^{-\frac{1}{2}}q^{-a'(s)}\frac{1-Lq^{a'(s)}t^{-\ell'(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}. \qquad \text{(B.22)}
$$

Then we have

$$
g_\lambda(q,t)\frac{P_\lambda(t^\rho, L_1t^{-\rho};q,t)}{P_{\lambda^\vee}(q^\rho, L_2q^{-\rho};t,q)} = \prod_{s\in\lambda}q^{a'(s)}t^{-\ell'(s)}\frac{1-L_1q^{-a'(s)}t^{\ell'(s)}}{1-L_2q^{a'(s)}t^{-\ell'(s)}}, \qquad L_1, L_2\in\mathbb{C}. \qquad \text{(B.23)}
$$

Note that

$$
\iota P_\lambda\left(t^{-\rho}, Lt^\rho;q,t\right) = P_\lambda\left(t^\rho, Lt^{-\rho};q,t\right), \qquad L\in\mathbb{C}. \qquad \text{(B.24)}
$$

If $L = t^{-N}$ with $N\in\mathbb{N}$, then $p_n(q^\lambda t^\rho, t^{-N-\rho}) = \sum_{i=1}^N q^{n\lambda_i}t^{n(\frac{1}{2}-i)}$ is the power sum symmetric polynomial in N variables $\{q^{\lambda_i}t^{\frac{1}{2}-i}\}_{1\le i\le N}$, hence $P_\lambda\left(t^\rho, t^{-N-\rho};q,t\right)$ reduces to the Macdonald symmetric polynomial in N variables. Therefore

$$
P_\lambda\left(t^\rho, t^{-N-\rho};q,t\right) = 0, \qquad \text{for} \quad \ell(\lambda) > N\in\mathbb{N}. \qquad \text{(B.25)}
$$

Note that

$$
\mathcal{W}_{\lambda,\mu}(q,t) := P_\lambda\left(t^\rho, t^{-N-\rho};q,t\right)P_\mu\left(q^\lambda t^\rho, t^{-N-\rho};q,t\right), \qquad N\in\mathbb{N}, \qquad \text{(B.26)}
$$

has a nice symmetry [15](Ch. VI.6):

$$\mathcal{W}_{\lambda,\mu}(q,t)=\mathcal{W}_{\mu,\lambda}(q,t). \tag{B.27}$$

When $L=0$ (the case of principal specialization),

$$\begin{aligned} P_\lambda\left(t^\rho;q,t\right)\prod_{s\in\lambda}(-1)q^{-a(s)}t^{\ell(s)} &= P_\lambda\left(t^{-\rho};q,t\right)=P_{\lambda^\vee}\left(-q^\rho;t,q\right)/g_\lambda(q,t)\\ &=\iota P_\lambda\left(t^\rho;q,t\right)=\iota P_{\lambda^\vee}\left(-q^{-\rho};t,q\right)/g_\lambda(q,t). \end{aligned} \tag{B.28}$$

Appendix C : Refined BPS State Counting

From the instanton expansion of Nekrasov's partition function,

$$Z_{Nek}=1+\sum_{k=1}^{\infty}\Lambda^k Z_k(Q_\alpha;q,t)\ , \tag{C.1}$$

we can compute the refined Gopakumar-Vafa integer invariant $N_\beta^{(j_L,j_R)}$ as follows. We expect the following multicover structure of the partition function

$$Z_{Nek}=\exp\left(\sum_{n=1}^{\infty}\frac{G(Q_\alpha^n,Q_B^n;q^n,t^n)}{n}\right)\ , \tag{C.2}$$

from the argument of Gopakumar-Vafa type. Assuming the scale parameter Λ is proportional to the Kähler parameter Q_B of the base space $\mathbf{P}^1$ of ALE fibration, we expand

$$G(Q_\alpha,Q_B;q,t)=\sum_{k=1}^{\infty}Q_B^k G_k(Q_\alpha;q,t)\ , \tag{C.3}$$

where

$$G_k(Q_\alpha;q,t)=\sum_{\{\ell_\alpha\}}\sum_{(j_L,j_R)}\frac{N_{k,\{\ell_\alpha\}}^{(j_L,j_R)}}{(q^{1/2}-q^{-1/2})(t^{1/2}-t^{-1/2})}\chi_{j_L}(u)\chi_{j_R}(v)\prod_{\alpha=1}^{N-1}Q_\alpha^{\ell_\alpha}\ , \tag{C.4}$$

and $\chi_j(x)$ is the irreducible character of $SU(2)$ with spin j. We have introduced the notations $u^2=t\cdot q$ and $v^2=q/t$. Comparing the coefficients of $\Lambda^k\sim Q_B^k$, up to $k=4$

we obtain

$$
\begin{aligned}
G_1(Q_\alpha;q,t) &= Z_1(Q_\alpha;q,t),\\
G_2(Q_\alpha;q,t) &= Z_2(Q_\alpha;q,t) - \frac{1}{2}\left(Z_1(Q_\alpha;q,t)\right)^2 - \frac{1}{2}Z_1(Q_\alpha^2;q^2,t^2),\\
G_3(Q_\alpha;q,t) &= Z_3(Q_\alpha;q,t) - Z_2(Q_\alpha;q,t)Z_1(Q_\alpha;q,t) + \frac{1}{3}\left(Z_1(Q_\alpha;q,t)\right)^3 - \frac{1}{3}Z_1(Q_\alpha^3;q^3,t^3),\\
G_4(Q_\alpha;q,t) &= Z_4(Q_\alpha;q,t) - Z_3(Q_\alpha;q,t)Z_1(Q_\alpha;q,t) - \frac{1}{2}\left(Z_2(Q_\alpha;q,t)\right)^2\\
&\quad + Z_2(Q_\alpha;q,t)\left(Z_1(Q_\alpha;q,t)\right)^2 - \frac{1}{4}\left(Z_1(Q_\alpha;q,t)\right)^4 - \frac{1}{2}Z_2(Q_\alpha^2;q^2,t^2)\\
&\quad + \frac{1}{4}\left(Z_1(Q_\alpha^2;q^2,t^2)\right)^2 .
\end{aligned}
\tag{C.5}
$$

There is a cancellation of $Z_1(Q_\alpha^4;q^4,t^4)$ in the computation of $G_4(Q_\alpha;q,t)$.

In [14] we reported some results for $SU(2)$ theory with no Chern-Simons coupling. This corresponds to the refined GV invariants for the local Hirzebruch surface of $\mathbf{F}_0$. For $SU(2)$ theory the expansion at instanton number k becomes

$$
G_k(Q_F;q,t) = \sum_{n=0}^{\infty}\sum_{(j_L,j_R)} \frac{N_{kB+nF}^{(j_L,j_R)}}{(q^{1/2}-q^{-1/2})(t^{1/2}-t^{-1/2})}\chi_{j_L}(u)\chi_{j_R}(v)v^{2k}Q_F^{k+n} , \tag{C.6}
$$

where Q_F is the Kähler parameter of the fiber $\mathbf{P}^1$. The analysis of the symmetry of Nekrasov's partition function made in section 2 instructs us to factor out $v^{2k}Q_F^k$ in computing $N_{kB+nF}^{(j_L,j_R)}$. Our results are

$$
N_{B+nF}^{(j_L,j_R)} = \delta_{j_L,0}\delta_{j_R,n+\frac{1}{2}} , \tag{C.7}
$$

for one instanton and

$$
\bigoplus_{(j_L,j_R)} N_{2B+nF}^{(j_L,j_R)}\,(j_L,j_R) = \bigoplus_{\ell=1}^{n}\bigoplus_{m=1}^{n-\ell+1}\left[\frac{m+1}{2}\right]\left(\frac{\ell-1}{2},\frac{3\ell+2m}{2}\right) , \tag{C.8}
$$

for two instantons.

We have computed the invariants of $SU(2)$ theory with the Chern-Simons coupling $m=1,2$, which are expected to give the refined GV invariants for local $\mathbf{F}_1$ and $\mathbf{F}_2$. It has been known that the GV invariants of $\mathbf{F}_0$ and $\mathbf{F}_2$ are simply related by a "shift" of the Kähler parameters. We have found this relation survives for the refined GV invariants up to instanton number 3. To describe the result neatly, let $G_k^{(m)}(Q_F;q,t)$ be the coefficients of the instanton expansion (C.3) for local $\mathbf{F}_m$. Then what we have checked is

$$
G_k^{(2)}(Q_F;q,t) = Q_F^k \cdot G_k^{(0)}(Q_F;q,t), \quad (1 \le k \le 3) , \tag{C.9}
$$

which implies $N^{(j_L,j_R)}_{kB+nF}$ for local $\mathbf{F}_0$ is the same as $N^{(j_L,j_R)}_{kB+(n+k)F}$ for local $\mathbf{F}_2$. We would like to stress this is a somewhat surprising result, since the refined GV invariants are not BPS protected quantities and they may jump under the deformation of complex structures[10]. For the GV invariants which are BPS protected, the agreement of the invariants may be explained by the fact that $\mathbf{F}_2$ is obtained from $\mathbf{F}_0$ by a deformation of complex structure[11]. However, for BPS nonprotected quantities it is not certain if the same argument applies. In any case what we have found supports the expectation that on noncompact Calabi-Yau manifold the refined GV invariants are actually invariant under the complex structure deformation, which is pointed out in [11].

For local $\mathbf{F}_1$ the invariants are qualitatively different from local $\mathbf{F}_0$ at one instanton. We have

$$N^{(j_L,j_R)}_{B+nF} = \delta_{j_L,0}\delta_{j_R,n} \ . \tag{C.10}$$

For $\mathbf{F}_0$ and $\mathbf{F}_2$ the right spin j_R at one instanton is always half-integer, while for $\mathbf{F}_1$ it is integer. However, at two instanton our computation shows that the refined GV invariants of $\mathbf{F}_1$ are related to $\mathbf{F}_0$ quite similarly to the relation between $\mathbf{F}_0$ and $\mathbf{F}_2$. We have checked that

$$G^{(1)}_{2k}(Q_F;q,t) = Q_F^k \cdot G^{(0)}_{2k}(Q_F;q,t), \quad (k=1) \ . \tag{C.11}$$

It has been pointed out that for even instanton number one may expect the GV invariants of local $\mathbf{F}_0$ and local $\mathbf{F}_1$ are related [7]. It is tempting to conjecture that the above relation is valid for any k.

For general values of the Chern-Simons coupling, our preliminary computation shows that the refined invariants have no simple relation to those of local $\mathbf{F}_{0,1,2}$. Even wrong the structure of $Spin(4)$ character seems lost in this region. This may be related to the fact that the five-dimensional theory is physically not well-defined for these Chern-Simons couplings.

For $SU(3)$ case the computation of the refined invariants gets more involved. The corresponding local toric Calabi-Yau geometry is the ALE fibration of A_2 type over $\mathbf{P}^1$ and we have two Kähler parameters $Q_1 := e^{-t_{F_1}}$ and $Q_2 := e^{-t_{F_2}}$, for the fiber. The

[10]However, for local CY the deformation of complex structure may not be well-defined, because of noncompactness of the total space.

[11]We thank Y. Konishi and S. Minabe for discussion on this issue.

instanton expansion takes the form

$$G_k(Q_i;q,t)=\sum_{n_1,n_2=0}^{\infty}\sum_{(j_L,j_R)}\frac{N^{(j_L,j_R)}_{\beta(n_1,n_2)}}{(q^{1/2}-q^{-1/2})(t^{1/2}-t^{-1/2})}\chi_{j_L}(u)\chi_{j_R}(v)v^{3k}Q_1^{k+n_1}Q_2^{k+n_2}\ , \tag{C.12}$$

where $\beta(n_1,n_2):=kB+n_1F_1+n_2F_2$ represents two cycles wrapping k times on the base space. The analysis of the symmetry of Nekrasov's partition function made in section 2 instructs us to factor out $v^{3k}(Q_1Q_2)^k$. At one instanton we found that the spin contents for the homology class $B+n_1F_1+n_2F_2$ are

$$(0,n_{\max})\oplus(0,n_{\max}-1)\oplus\cdots\oplus(0,|n_1-n_2|)\ , \tag{C.13}$$

where $n_{\max}:=\max(n_1,n_2)$. We note that the left spin always vanishes at one instanton. When $n_1=0$ or $n_2=0$ the geometry reduces to local $\mathbf{F}_1$ and the above result is consistent with the refined GV invariants of local $\mathbf{F}_1$. At two instanton since we cannot find any simple rule for the refined GV invariants, let us present a short list of our computation. When $n_1=0$ or $n_2=0$, the result is again consistent with (C.8) in view of the relation (C.11).

(n_1, n_2)	spin contents
$(1,0),(0,1),(1,1)$	$\emptyset$
$(2,0),(0,2)$	$(0,\frac{5}{2})$
$(2,1),(1,2)$	$(0,\frac{5}{2})\oplus(0,\frac{3}{2})$
$(3,0),(0,3)$	$(\frac{1}{2},4)\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$
$(2,2)$	$(0,\frac{7}{2})\oplus 2(0,\frac{5}{2})\oplus 2(0,\frac{3}{2})\oplus 2(0,\frac{1}{2})$
$(3,1),(1,3)$	$(\frac{1}{2},4)\oplus(\frac{1}{2},3)\oplus 2(0,\frac{7}{2})\oplus 3(0,\frac{5}{2})\oplus(0,\frac{3}{2})$
$(4,0),(0,4)$	$(1,\frac{11}{2})\oplus(\frac{1}{2},5)\oplus(\frac{1}{2},4)\oplus 2(0,\frac{9}{2})\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$
$(3,2),(2,3)$	$(\frac{1}{2},4)\oplus(\frac{1}{2},3)\oplus(\frac{1}{2},2)$ $\oplus(0,\frac{9}{2})\oplus 3(0,\frac{7}{2})\oplus 5(0,\frac{5}{2})\oplus 4(0,\frac{3}{2})\oplus 2(0,\frac{1}{2})$
$(4,1),(1,4)$	$(1,\frac{11}{2})\oplus(1,\frac{9}{2})\oplus(\frac{1}{2},5)\oplus 3(\frac{1}{2},4)\oplus(\frac{1}{2},3)$ $\oplus 3(0,\frac{9}{2})\oplus 5(0,\frac{7}{2})\oplus 3(0,\frac{5}{2})\oplus(0,\frac{3}{2})$
$(5,0),(0,5)$	$(\frac{3}{2},7)\oplus(1,\frac{13}{2})\oplus(1,\frac{11}{2})\oplus 2(\frac{1}{2},6)\oplus(\frac{1}{2},5)\oplus(\frac{1}{2},4)$ $\oplus 2(0,\frac{11}{2})\oplus 2(0,\frac{9}{2})\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$
$(3,3)$	$(\frac{1}{2},5)\oplus 2(\frac{1}{2},4)\oplus 2(\frac{1}{2},3)\oplus 2(\frac{1}{2},2)\oplus 2(\frac{1}{2},1)$ $\oplus(0,\frac{11}{2})\oplus 3(0,\frac{9}{2})\oplus 6(0,\frac{7}{2})\oplus 8(0,\frac{5}{2})\oplus 8(0,\frac{3}{2})\oplus 6(0,\frac{1}{2})$

Appendix D : q-Dunkl Operator Realization for the Refined Topological Vertex

In this appendix, we use the Macdonald polynomials $P_\lambda^N(x;q,t)$ in the finite number of variables $x=(x_1,x_2,\cdots,x_N)$ with setting $x_{N+1}=x_{N+2}=\cdots=0$. Here we assume that $|q|$, $|t|>1$, and define the following refined topological vertex (without framing factor)

$$\begin{aligned}
V_{\mu\lambda}{}^{\nu} &:= \lim_{N\to\infty}\sum_\sigma \iota P^N_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ P^N_{\nu/\sigma}(q^\lambda t^\rho;q,t)P^N_\lambda(t^\rho;q,t)v^{|\sigma|},\\
V_{\mu}{}^{\lambda\nu} &:= \lim_{N\to\infty}\sum_\sigma P^N_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^\rho;t,q)\ \iota P^N_{\nu/\sigma}(q^\lambda t^\rho;q,t)P^N_{\lambda^\vee}(-q^\rho;t,q)v^{|\sigma|}=\iota V_{\mu\lambda}{}^{\nu}.
\end{aligned}\tag{D.1}$$

These also reproduce Nekrasov's partition function.

Let Y_i, $(i=1,\cdots,N)$ be the q-Dunkl operator [37] [38] acting on the variables x_i, $(i=1,\cdots,N)$;

$$\begin{aligned}
Y_i(x) &= t^{-\frac{N}{2}}T_iT_{i+1}\cdots T_{N-1}\omega T_1^{-1}\cdots T_{i-1}^{-1},\\
T_i &= t^{\frac12}+t^{-\frac12}\frac{1-tx_i/x_{i+1}}{1-x_i/x_{i+1}}(s_i-1),
\end{aligned}\tag{D.2}$$

where

$$s_i=(i,i+1),\qquad \omega=\tau_N s_{N-1}\cdots s_1,\qquad \tau_N(x_i)=q^{\delta_{i,N}}x_i.\tag{D.3}$$

They commute with each other,

$$[Y_i(x),Y_j(x)]=0,\tag{D.4}$$

and the Macdonald polynomials are eigenfunctions of any symmetric operator f in them:

$$f(Y_1(x),\cdots,Y_N(x))P^N_\lambda(x;q,t)=f(q^{\lambda_1}t^{\frac12-1},\cdots,q^{\lambda_N}t^{\frac12-N})P^N_\lambda(x;q,t).\tag{D.5}$$

Let $\widetilde{Y}_i(x)$ be the dual q-Dunkl operator which is given by replacing q with t in $Y_i(x)$, i.e.

$$f(\widetilde{Y}_1(x),\cdots,\widetilde{Y}_N(x))P^N_\lambda(x;t,q)=f(t^{\lambda_1}q^{\frac12-1},\cdots,t^{\lambda_N}q^{\frac12-N})P^N_\lambda(x;t,q).\tag{D.6}$$

Note that $Y_i(x)$ and $\widetilde{Y}_i(x)$ may not commute with each other.

Using these (dual) q-Dunkl operators, our vertices in (D.1) are written as follows:

$$\begin{aligned}
V_{\mu\lambda}{}^{\nu} &= \lim_{N\to\infty}\sum_\sigma v^{|\sigma|}P^N_{\nu/\sigma}(-Y(x);q,t)\ \omega_{q,t}\left(\iota P^N_{\mu^\vee/\sigma^\vee}(\widetilde{Y}(x);t,q)P^N_{\lambda^\vee}(x;t,q)\right)|_{x=t^\rho},\\
V_{\mu}{}^{\lambda\nu} &= \lim_{N\to\infty}\sum_\sigma v^{|\sigma|}P^N_{\mu^\vee/\sigma^\vee}(-\widetilde{Y}(x);t,q)\ \omega_{q,t}\left(\iota P^N_{\nu/\sigma}(Y(x);q,t)P^N_\lambda(-x;q,t)\right)|_{x=q^\rho}
\end{aligned}\tag{D.7}$$

Here $\omega_{q,t}$ is the involution in (B.13).

Therefore the summation in the Young diagrams in Nekrasov's formula is formally performed by using these q-Dunkl operators. For example, the $SU(2)$ partition function in (8.3) is

$$\begin{aligned}\widetilde{Z} = \lim_{N\to\infty} \Pi_0\left(-Q_1 Y(x), \widetilde{Y}(z)\right)^{-1} \Pi_0\left(-Q_2 Y(w), \widetilde{Y}(y)\right)^{-1} \\ \times \Pi_0(-\Lambda x, y)\ \Pi_0(-\Lambda z, w)|_{x=z=t^\rho, y=w=q^\rho}. \end{aligned} \tag{D.8}$$

In the $SU(N_c)$ case, let

$$\begin{aligned} D_0 &:= \prod_{\alpha=1}^{N_c} \Pi_0\left(-\Lambda x^\alpha, y^\alpha\right), \\ D_\alpha &:= \prod_{\beta=\alpha+1}^{N_c} \Pi_0\left(-Q_{\alpha,\beta} Y(x^\alpha), \widetilde{Y}(x^\beta)\right)^{-1} \Pi_0\left(-Q_{\beta,\alpha} Y(y^\beta), \widetilde{Y}(y^\alpha)\right)^{-1}, \quad 0 < \alpha < N_c, \end{aligned} \tag{D.9}$$

and $D'_\alpha := D_\alpha \omega_{t,q}^{x^\alpha} \omega_{q,t}^{y^\alpha}$, then the $SU(N_c)$ partition function in (8.7) with $m=0$ is

$$\begin{aligned} \widetilde{Z}_0 &= \sum_{\lambda_\alpha,\mu_\alpha,\nu_\alpha} V_{\bullet\lambda_1}{}^{\mu_1} V_{\mu_1\lambda_2}{}^{\mu_2} \cdots V_{\mu_{N_c-2}\lambda_{N_c-1}}{}^{\mu_{N_c-1}} V_{\mu_{N_c-1}\lambda_{N_c}}{}^{\bullet} \\ &\qquad \times V_{\bullet}{}^{\lambda_{N_c}\nu_{N_c-1}} V_{\nu_{N_c-1}}{}^{\lambda_{N_c-1}\nu_{N_c-2}} \cdots V_{\nu_2}{}^{\lambda_2\nu_1} V_{\nu_1}{}^{\lambda_1\bullet} \\ &\qquad \times \prod_{\alpha=1}^{N_c} Q_{B^\alpha}^{|\lambda_\alpha|} \prod_{\alpha=1}^{N_c-1} v^{-|\mu_\alpha|-|\nu_\alpha|} Q_{\alpha,\alpha+1}^{|\mu_\alpha|} Q_{\alpha+1,\alpha}^{|\nu_\alpha|} \\ &= D'_{N_c-1} \cdots D'_2 D_1 D_0|_{x^\alpha=t^\rho, y^\alpha=q^\rho}. \end{aligned} \tag{D.10}$$

Since $\omega_{t,q}^x \omega_{q,t}^y \Pi_0(x,y) = \Pi_0(x,y)$, we have the following q-Dunkl operator realization for Nekrasov's formula

$$\widetilde{Z}_0 = D_{N_c-1} \cdots D_2 D_1 D_0|_{x^\alpha=t^\rho, y^\alpha=q^\rho}. \tag{D.11}$$

Appendix E : Notations and identities for Partitions

For each square $s=(i,j)$ in the Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, we define

$$a_\lambda(s) := \lambda_i - j, \quad \ell_\lambda(s) := \lambda_j^\vee - i, \quad a'(s) := j-1, \quad \ell'(s) := i-1, \tag{E.1}$$

where $\lambda_j^\vee$ denotes the conjugate (dual) diagram. They are called arm length, leg length, arm colength and leg colength, respectively. The hook length $h_\lambda(s)$ and the content $c(s)$ at s are given by

$$h_\lambda(s) := a_\lambda(s) + \ell_\lambda(s) + 1, \quad c(s) := a'(s) - \ell'(s). \tag{E.2}$$

The weight $|\lambda|$ and $||\lambda||^2$ are

$$|\lambda| := \sum_i \lambda_i, \qquad ||\lambda||^2 := \sum_i {\lambda_i}^2 = 2 \sum_{s\in\lambda} (a(s) + \frac{1}{2}). \tag{E.3}$$

We also need the following integer

$$n(\lambda) := \sum_{s\in\lambda} \ell'(s) = \sum_{i=1}^{\infty} (i-1)\lambda_i = \frac{1}{2} \sum_{i=1}^{\infty} \lambda_i^\vee(\lambda_i^\vee - 1) = \sum_{s\in\lambda} \ell_\lambda(s) \ . \tag{E.4}$$

Similarly, we have

$$n(\lambda^\vee) := \sum_{s\in\lambda} a'(s) = \sum_{s\in\lambda} a_\lambda(s) \ . \tag{E.5}$$

They are related to the integer $\kappa(\lambda)$ as follows:

$$\kappa(\lambda) := 2 \sum_{s\in\lambda} (j-i) = 2(n(\lambda^\vee) - n(\lambda)) = |\lambda| + \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i) \ . \tag{E.6}$$

Note that, since $\{\lambda_i - j\}_{j=1}^{\lambda_i} = \{j-1\}_{j=1}^{\lambda_i}$,

$$\sum_{(i,j)\in\lambda} f(\lambda_i - j,\ i) = \sum_{(i,j)\in\lambda} f(j-1,\ i), \tag{E.7}$$

for any function f. Also since

$$\{\lambda_i - j\}_{j=1}^{\mu_i} \cup \{-\mu_i + j - 1\}_{j=1}^{\lambda_i} = \{j\}_{j=-\mu_i}^{\lambda_i - 1} = \{\lambda_i - j\}_{j=1}^{\lambda_i} \cup \{-\mu_i + j - 1\}_{j=1}^{\mu_i}, \tag{E.8}$$

we have

$$\sum_{(i,j)\in\mu} (\lambda_i - j) - \sum_{(i,j)\in\lambda} (\mu_i - j + 1) = \sum_{(i,j)\in\lambda} (\lambda_i - j) - \sum_{(i,j)\in\mu} (\mu_i - j + 1). \tag{E.9}$$

We list their relations:

$\lvert\lambda\rvert, \lvert\lvert\lambda\rvert\rvert^2, n, \kappa$	$\sum_i$	$\sum_{(i,j)\in\lambda}$	$\sum_{s\in\lambda}$

$$
\begin{array}{ccccccc}
|\lambda| & = & \sum_i \lambda_i & = & \sum_{(i,j)\in\lambda} 1 & = & \sum_{s\in\lambda} 1, \\
|| & & & & & & \\
|\lambda^\vee| & = & \sum_j \lambda_j^\vee . & & & & \\
\frac{1}{2}||\lambda||^2 & = & \frac{1}{2}\sum_i {\lambda_i}^2 & = & \sum_{(i,j)\in\lambda}(\lambda_i - j + \frac{1}{2}) & = & \sum_{s\in\lambda}\left(a(s)+\frac{1}{2}\right). \\
\frac{1}{2}||\lambda^\vee||^2 & = & \frac{1}{2}\sum_j \lambda_j^{\vee 2} & = & \sum_{(i,j)\in\lambda}(\lambda_j^\vee - i + \frac{1}{2}) & = & \sum_{s\in\lambda}\left(\ell(s)+\frac{1}{2}\right),
\end{array}
$$

$$
\begin{array}{ccccccc}
n(\lambda) & = & \sum_i (i-1)\lambda_i & = & \sum_{(i,j)\in\lambda}(i-1) & = & \sum_{s\in\lambda} \ell'(s), \\
|| & & & & & & \\
\frac{1}{2}(||\lambda^\vee||^2 - |\lambda^\vee|) & = & \frac{1}{2}\sum_j \lambda_j^\vee(\lambda_j^\vee - 1) & = & \sum_{(i,j)\in\lambda}(\lambda_j^\vee - i) & = & \sum_{s\in\lambda} \ell(s). \\
n(\lambda^\vee) & = & \sum_j (j-1)\lambda_j^\vee & = & \sum_{(i,j)\in\lambda}(j-1) & = & \sum_{s\in\lambda} a'(s), \\
|| & & & & & & \\
\frac{1}{2}(||\lambda||^2 - |\lambda|) & = & \frac{1}{2}\sum_i \lambda_i(\lambda_i - 1) & = & \sum_{(i,j)\in\lambda}(\lambda_i - j) & = & \sum_{s\in\lambda} a(s).
\end{array}
$$

$$
\begin{array}{ccccccc}
\frac{1}{2}\kappa(\lambda) & = & \frac{1}{2}\sum_i \lambda_i(\lambda_i + 1 - 2i), & & & & \sum_{s\in\lambda} c(s), \\
|| & & & & & & || \\
n(\lambda^\vee) - n(\lambda) & = & \sum_i i\,(\lambda_i^\vee - \lambda_i) & = & \sum_{(i,j)\in\lambda}(j-i) & = & \sum_{s\in\lambda}(a'(s) - \ell'(s)), \\
|| & & & & & & \\
\frac{1}{2}(||\lambda||^2 - ||\lambda^\vee||^2) & = & \frac{1}{2}\sum_i \left({\lambda_i}^2 - \lambda_i^{\vee 2}\right) & = & \sum_{(i,j)\in\lambda}(\lambda_i - \lambda_j^\vee + i - j) & = & \sum_{s\in\lambda}(a(s) - \ell(s)). \\
n(\lambda^\vee) + n(\lambda) + |\lambda| & = & \sum_i \left(i - \frac{1}{2}\right)(\lambda_i + \lambda_i^\vee) & = & \sum_{(i,j)\in\lambda}(i+j-1) & = & \displaystyle\sum_{s\in\lambda}(a'(s) + \ell'(s) + 1), \\
|| & & & & & & \\
\frac{1}{2}(||\lambda||^2 + ||\lambda^\vee||^2) & = & \frac{1}{2}\sum_i \left({\lambda_i}^2 + \lambda_i^{\vee 2}\right) & = & \displaystyle\sum_{(i,j)\in\lambda}(\lambda_i + \lambda_j^\vee - i - j + 1) & = & \displaystyle\sum_{s\in\lambda}(a(s) + \ell(s) + 1), \\
 & & & & & & || \\
 & = & \frac{1}{2}\sum_i \lambda_i(\lambda_i - 1 + 2i), & & & & \sum_{s\in\lambda} h(s).
\end{array}
$$

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